

# The theory of superqubits

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Superqubits are the minimal supersymmetric extension of qubits. In this paper we investigate in detail their unusual properties with emphasis on their potential role in (super)quantum information theory and foundations of quantum mechanics. We propose a partial solution to the problem of negative transition probabilities that appear in the theory and has been previously reported in [15]. The modification does not affect the performance of supersymmetric entangled states in the CHSH game – superqubits provide resources more nonlocal than it is allowed by ordinary quantum mechanics.

Keywords: supersymmetry, superqubit, Tsirelson’s bound, CHSH

It is a widely accepted fact nowadays that quantum mechanics is a qualitatively different theory compared to classical mechanics. Quantum mechanics provides resources such as entangled pure or mixed states that are impossible to simulate in classical physics. But an interesting point was raised by Popescu and Rohrlich in [4] almost two decades ago. They asked why quantum mechanics could not have been even more nonlocal than it actually is. First, they recalled an earlier result by Tsirelson [2] who showed that in a certain version of Bell’s inequalities [1], known as the CHSH inequality [3], no quantum-mechanical state can violate the inequality more than a maximally entangled state. So their question was: Going beyond Tsirelson’s bound, does it clash with other established principles of physics, namely the impossibility of superluminal communication (the so-called no-signalling condition)? Surprisingly, the answer is no and there exists a gap between Tsirelson’s bound and unhealthy theories allowing faster-than-light communication. Henceforth, quantum mechanics is not the maximally nonlocal theory it could have been. That immediately raises another question whether a consistent theory inhabiting the gap could be constructed or even realized in Nature. A considerable effort has been recently spent on investigating the consequences of such a superquantum theory [6, 7, 9] and a number of results showed that if Tsirelson’s bound is crossed certain entropic quantities valid in quantum mechanics would become invalid [10–13]. This would have significant implications for the fields of quantum information theory and foundations of physics [8].

An important tool in these investigations is a hypothetical resource called a nonlocal box also proposed in [4]. It is a superquantum resource performing strictly better than quantum mechanics while still respecting the no-signalling condition. Nonlocal boxes rule over the whole gap in a sense that their decohered version can approximate any theory between quantum mechanics and the no-signalling world. They are, however, purely mathematical constructs with no links to even hypothetical physical theories.

The object of study in this work is called a superqubit. It was first introduced in [14] and its role in the CHSH game was investigated for the first time in [15]. Put simply, superqubits are the minimal supersymmetric extension of qubits where the main role is played by the orthosymplectic Lie superalgebra  $osp(1|2)$  (more precisely, one of its real forms). The  $osp(1|2)$  algebra has been extensively studied in the past [26, 29–33] as one of the most important example of a Lie superalgebras [16–23]. On the physical side, the main motivation for studying supersymmetry comes from high energy physics where it is the leading candidate for physics beyond Standard Model. Apart from high-energy physics there exists at least two proposals from condensed matter physics where supersymmetry, in particular the family of orthosymplectic Lie superalgebras  $osp(p|q)$ , plays a key role [34, 35].

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The result of [15] suggests that the supersymmetric extension of quantum mechanics based on superqubits may be a candidate for a superquantum theory that lies in the gap between ordinary quantum theory and PR boxes. This is due to the violation of Tsirelson's bound reported there. An unfortunate consequence is the presence of negative transition probabilities. They never appear in the actual calculation leading to the result but one would like to avoid them entirely. Here we propose a partial solution to this problem that works for single superqubits but is not sufficient for multipartite states.

This paper has a multiple purpose: (i) to anchor the notion of superqubits on a firm mathematical footing that has its roots in the theory of Lie superalgebras and related structures, (ii) to systematically develop the rules for calculating with superqubits in the same way quantum information theorists deal with qubits and, (iii) to offer a solution to the problem of occurrence of negative probabilities. This is an issue first encountered in [15] and we offer a partial solution by means of compactification of the superqubit space. As a result, the problem of negative probabilities disappears for single superqubits but not for multi-superqubit states where more work has to be done. An important consequence is that we violate Tsirelson's bound less ( $p_{win}^{sqbit} \simeq 0.8647$ ) than reported in [15].

There are three main sections in this paper. In Section I we summarize mostly well-known facts about  $\mathbb{Z}_2$ -graded vector spaces, superalgebras, Lie superalgebras and supermodules. Section II introduces superqubits in a manner slightly different from the original article [14] and heavily relies on the previous section. It brings some additional, perhaps less known or even novel, details about the used superstructures that help to formalize the notion of a superqubit. Both sections are full of remarks that provide further explanations or point to less related issues. In Section III we focus on the physical side of bipartite superqubit states and investigate its performance in the CHSH game. We conclude with a number of open questions.

## I. BACKGROUND ON LIE SUPERALGEBRAS AND RELATED STRUCTURES

**Definition 1.** Let  $W = W^{[0]} \oplus W^{[1]}$  be a finite-dimensional  $\mathbb{Z}_2$ -graded linear vector space where the grading structure is isomorphic to  $\mathbb{Z}_2$ . When

$$\begin{aligned} \dim W^{[0]} &= p \\ \dim W^{[1]} &= q \end{aligned}$$

we will write  $W = W^{p|q}$  to indicate  $\dim W^{p|q} = p + q$ .

An element  $w$  of the vector space is called homogeneous if  $w \in W^{[i]}$ . The degree of a homogeneous element is defined  $\deg w \equiv |w| = i \in \mathbb{Z}_2$ . The zero (one) degree elements are called even (odd).

The property that makes  $\mathbb{Z}_2$ -graded vector spaces different from ordinary vector spaces is that the tensor product obeys the grading structure:

$$(V \otimes W)^{[k]} = \bigoplus_{k=l \oplus m} V^{[l]} \otimes W^{[m]},$$

where  $\oplus$  stands for addition modulo two.

*Remark.* It is quite common in supermathematics that there exists several different terms for the same structure. We will encounter this situation in the following text and the alternatives will be shown in parenthesis. The first example is the very notion of degree. The other options are parity and grade. Some authors insist on distinction between grade and degree [17]. In the present work these two terms will be used interchangeably.

**Definition 2.** A superalgebra is a  $\mathbb{Z}_2$ -graded vector space equipped with a product  $[\cdot] : W \times W \rightarrow W$  called the supercommutator (graded commutator)

$$[w, v] = wv - (-)^{|w||v|}vw \quad (1)$$

defined for all  $w, v \in W$ .

A supercommutative (super)algebra is a superalgebra where

$$[w, v] = 0$$

for all  $w, v \in W$ .

**Example.** A complex Grassmann algebra  $\mathcal{C}\Lambda_N(W)$  of order  $N$  is a complex  $\mathbb{Z}_2$ -graded vector space  $W$  equipped with a supercommutative multiplication  $[\cdot] : W \times W \mapsto W$  called the wedge product, where

$$[w, v] = (-1)^{|w||v|}[v, w] \quad (2)$$

holds for all homogeneous  $w, v \in W$ . A Grassmann algebra  $\mathcal{C}\Lambda_N(W)$  is generated by  $N$  anti-commuting generators  $\{\eta^i\}_{i=1}^N$ .

*Remark.* The Grassmann algebra  $\mathcal{C}\Lambda_N(W)$  is an example of a supercommutative algebra and is isomorphic to the exterior algebra  $\wedge^N(W)$ . By linearity of the wedge product the supercommutator can be extended to non-homogeneous elements  $w \in W^{p|q}$ . The Grassmann algebra of order  $N$  has a direct sum structure

$$\mathcal{C}\Lambda_N = \bigoplus_{k=0}^N \mathcal{C}\Lambda_N^k,$$

where  $\dim \mathcal{C}\Lambda_N^k = \binom{N}{k}$ . The dimension of the Grassmann algebra  $\mathcal{C}\Lambda_N$  is  $2^N$  and contains a unit element in  $\mathcal{C}\Lambda_N^0 \equiv \mathbb{C}$ . Note that throughout this work we will consider only finite-dimensional Grassmann algebras. Also, we will abbreviate  $\mathcal{C}\Lambda_N \equiv \mathcal{C}\Lambda_N(W^{p|q})$  and  $\mathcal{C}\Lambda_{N,i} \equiv \mathcal{C}\Lambda_N(W^{[i]})$ .

**Definition 3.** An arbitrary element  $\zeta \in \mathcal{C}\Lambda_N$  is called a supernumber and can be uniquely decomposed as  $\zeta = \zeta_e + \zeta_o$  where  $\zeta_e \in \mathcal{C}\Lambda_{N,0}$  and  $\zeta_o \in \mathcal{C}\Lambda_{N,1}$ . The general form of an even and odd supernumber reads

$$\zeta_e = z_0 + \sum_{k \in \mathbb{N}_e} \sum_{m=1}^{\binom{N}{k}} \frac{1}{k!} z_I^{(m)} \eta^I = z_0 + \sum_{k \in \mathbb{N}_e} \sum_{m=1}^{\binom{N}{k}} z^{(m)} \eta \quad (3)$$

$$\zeta_o = \sum_{k \in \mathbb{N}_o} \sum_{m=1}^{\binom{N}{k}} \frac{1}{k!} z_I^{(m)} \eta^I = \sum_{k \in \mathbb{N}_o} \sum_{m=1}^{\binom{N}{k}} z^{(m)} \eta, \quad (4)$$

where  $z_0, z_I^{(m)} \in \mathbb{C}$ ,  $\mathbb{N}_e(\mathbb{N}_o)$  is a subset of even (odd) integers  $\mathbb{N}_e = \{2n; 1 \leq n \leq \lfloor \frac{N}{2} \rfloor\}$  ( $\mathbb{N}_o = \{2n-1; 1 \leq n \leq \lfloor \frac{N+1}{2} \rfloor\}$ ) and the multiindex  $I$  is defined as  $I = [i_1 \dots i_k]$  where  $\eta^I = \eta^{i_1} \dots \eta^{i_k}$  is a product of  $k$  Grassmann generators.

Furthermore, we will call even Grassmann numbers of grade zero and odd Grassmann numbers of grade one where the grade will be denoted by vertical lines:  $|\zeta_e| \stackrel{\text{df}}{=} 0$  and  $|\zeta_o| \stackrel{\text{df}}{=} 1$ .

Note that we sum over  $I$  but since  $z_I^{(m)}$  is a completely antisymmetric tensor we set  $I = 1 \dots k$  and so  $z^{(m)} = z_I^{(m)}$  and  $\eta = \eta^I$  on the RHS of the above equations.

Every complex superalgebra is associated with at least two types of antilinear automorphisms.

**Definition 4.** Let  $W$  be a complex superalgebra (not necessarily supercommutative). There exists a mapping  $*$  :  $W \mapsto W$  satisfying

$$(wv)^* = v^* w^* \quad (5)$$

$$(w^*)^* = w \quad (6)$$

for all  $w, v \in W$ . The star operation is therefore an involution. The second automorphism is a grade involution  $\# : W \mapsto W$  whose properties are

$$(wv)^\# = (-1)^{|w||v|} v^\# w^\# = w^\# v^\# \quad (7a)$$

$$(w^\#)^\# = (-1)^{|w|} w. \quad (7b)$$

The RHS of the first line follows from the properties of the supercommutative product (Eq. (2)).

Both mappings reduce to ordinary complex conjugation for complex numbers. When  $W = \mathcal{C}\Lambda_N$  the star map is frequently used in calculations of fermion path integrals in QFT [44]. For us, however, will be important the hash map since it naturally appears for the Lie superalgebras studied here. For more details in the connection with a Grassmann algebra see [16, 33].

**Definition 5.** A finite-dimensional  $\mathbb{Z}_2$ -graded linear vector space  $W^{p|q}$  is called the Lie superalgebra if it is equipped with a bilinear non-associative product  $[\cdot] : W \times W \mapsto W$  satisfying [18]

$$[w, v] = -(-1)^{|w||v|} [v, w] \quad (8)$$

$$0 = (-1)^{|w||u|} [w, [v, u]] + (-1)^{|v||w|} [v, [u, w]] + (-1)^{|u||v|} [u, [w, v]] \quad (9)$$

for all  $u, v, w \in W$ . The graded commutator Eq. (1) satisfies the above conditions.

**Definition 6.** Let  $W^{p|q}$  be a  $\mathbb{Z}_2$ -graded linear vector space. A linear operator  $A \in \text{End}(W^{p|q})$  is said to be even (bosonic) if it is grade-preserving

$$A(W^{[i]}) = W^{[i]}$$

and we write  $|A| = 0$ . Similarly,  $A$  is called odd (fermionic) if it is grade-reversing

$$A(W^{[i]}) = W^{[i \oplus 1]}$$

( $|A| = 1$ ). The symbol  $\oplus$  denotes addition modulo two.

**Example.** The general linear Lie superalgebra is defined as  $gl(p|q; \mathbb{K}) = \text{End}(W^{p|q})$  and consists of all even and odd linear operators whose entries lie in  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ .

The  $gl(p|q; \mathbb{K})$  plays the same role in the representation theory of Lie superalgebras as the general linear algebra  $gl(p, \mathbb{K})$  [26] (recall that we consider only finite-dimensional vector spaces in this work). In particular, a representation  $\varrho$  of a Lie superalgebra  $V$  is defined as an even homomorphism

$$\varrho : V \mapsto gl(p|q; \mathbb{K}).$$

The introduction of  $\mathbb{Z}_2$ -graded complex vector spaces such as  $gl(p|q; \mathbb{C})$  dramatically enriches the structures we can define and study. Following [26] we assume the vector space to be equipped with a non-degenerate Hermitian form (denoted by  $\langle | \rangle$ ) such that the even and odd subspace of  $W^{p|q}$  are orthogonal with respect to it:  $\langle W^{p|0} | W^{0|q} \rangle = 0$ . There exists a thoroughly studied mapping  $\dagger : W^{p|q} \mapsto W^{p|q}$  called the adjoint operator with well documented properties upon which the mathematics of quantum mechanics is built. But  $\mathbb{Z}_2$ -graded spaces offer more than this.

**Definition 7.** Let  $A, B$  be homogeneous elements of  $gl(p|q; \mathbb{C})$ . There exists a mapping  $\dagger : \text{End}(W^{p|q}) \mapsto \text{End}(W^{p|q})$  called the grade adjoint defined by

$$\langle A^\dagger w | v \rangle = (-)^{|A||w|} \langle w | Av \rangle, \quad (10)$$

valid for all homogeneous  $w, v \in W^{p|q}$ . The grade adjoint has the following properties:

$$(aA + bB)^\dagger = \bar{a}A^\dagger + \bar{b}B^\dagger, \quad (11)$$

$$(AB)^\dagger = (-)^{|A||B|} B^\dagger A^\dagger, \quad (12)$$

$$(A^\dagger)^\dagger = (-)^{|A|} A, \quad (13)$$

where  $a, b \in \mathbb{C}$  and the bar denotes complex conjugation. Note that  $A^\dagger = A^\dagger$  for even linear operators.

To proceed we define a grade star representation of a Lie superalgebra  $V$  as

$$\varrho(A^\dagger) = \varrho(A)^\dagger \quad (14)$$

valid for all  $A \in V$ . Let  $W^{p|q}$  be the representation space again equipped with a non-degenerate Hermitian form. But this time we ask for this form to be positive semidefinite  $\langle w_e^i | w_e^i \rangle \geq 0$ ,  $\langle w_o^j | w_o^j \rangle \geq 0$ . Therefore  $W^{p|q} = W^{p|0} \oplus W^{0|q}$  is a  $\mathbb{Z}_2$ -graded Hilbert space where  $|w_e^i| = 0$  and  $|w_o^j| = 1$  is even and odd element, respectively, and  $\langle w_o^i | w_e^j \rangle = 0$  for all  $i, j$ .

The problem with the grade adjoint appears when we consider a tensor product of two grade star representations. It can be shown [25] that the resulting vector space is not a Hilbert space and we will find an explicit example when studying a representation of a certain subalgebra of  $gl(p|q; \mathbb{C})$ . Note that this does not happen for the ordinary adjoint as can be readily demonstrated in the case of two fermions. A single fermion space is clearly a Hilbert space:  $\langle vac | f f^\dagger | vac \rangle = 1$  where  $f^\dagger | vac \rangle = |1\rangle$  is an odd state. A two-fermion state  $|11\rangle_{12} = f_1^\dagger f_2^\dagger | vac \rangle$  occupies a Hilbert space as well

$$\dagger : f_1^\dagger f_2^\dagger \mapsto f_2 f_1 \quad (15)$$

and so  ${}_{21}\langle 11 | 11 \rangle_{12} = \langle 1 | 1 \rangle_2 \langle 1 | 1 \rangle_1 = 1$  using  $f_i f_j^\dagger = -f_j^\dagger f_i$  for  $i \neq j$ . Note that Eq. (15) should be written as  $\dagger : f_1^\dagger \otimes f_2^\dagger \mapsto f_2 \otimes f_1$  where  $\otimes$  is an antisymmetric tensor product. We will abbreviate if there is no chance of confusion.

**Definition 8.** Let  $A$  be a superalgebra over  $\mathbb{R}$  or  $\mathbb{C}$ . The left  $A$ -supermodule is  $\mathbb{Z}_2$ -graded vector space  $W$  endowed with a left multiplication  $A \times W \mapsto W$ . Similarly, for the right  $A$ -supermodule we have a right multiplication  $W \times A \mapsto W$ . If the superalgebra  $A$  is supercommutative, both multiplications are related by

$$av = (-)^{|a||v|} va, \quad (16)$$

for all  $a \in A$  and  $v \in W$ .

All graded vector spaces we encounter here will be finite-dimensional. Hence, all linear operators acting on  $W^{p|q}$  can be represented by matrices of the block form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (17)$$

where  $\dim X = p + q$ . So, for example, the submatrix  $C$  is a rectangular block with  $q$  rows and  $p$  columns. The matrix  $X$  has entries in either  $\mathbb{R}$  or  $\mathbb{C}$ .

But the basic structure we will work with is a left  $A$ -supermodule where  $A$  is a complex Grassmann algebra of order  $N$ . Elements of the left  $A$ -supermodule are represented by matrices whose entries are Grassmann numbers and we will call such a matrix a supermatrix denoting it  $X(\mathcal{C}\Lambda_N)$ . A supermatrix itself can be written as a sum of an even and odd supermatrix

$$X(\mathcal{C}\Lambda_N) = X(\mathcal{C}\Lambda_N)_0 + X(\mathcal{C}\Lambda_N)_1. \quad (18)$$

An even supermatrix  $X(\mathcal{C}\Lambda_N)_0$  is a Grassmann-valued matrix of the form Eq. (17) where the entries of the matrices  $A$  and  $D$  are even supernumbers and the entries of  $B$  and  $C$  are odd supernumbers. For an odd supermatrix  $X(\mathcal{C}\Lambda_N)_1$  the degree of Grassmann entries of the submatrices  $A, B, C$  and  $D$  is reversed.

The set of supermatrices forms a vector space but some additional structures are present. We will not discuss them here [24] but since the main topic of this paper is actually a certain subspace of even supermatrices we will study its properties in great detail. But before we do so it is necessary to list some basic facts from superlinear algebra. The supertranspose of  $X$  is defined

$$X^{ST} \stackrel{\text{df}}{=} \begin{pmatrix} A^T & (-)^{|X|} C^T \\ -(-)^{|X|} B^T & D^T \end{pmatrix}, \quad (19)$$

where  $M^T$  denotes the transposition of a matrix  $M$  in a given basis. We pinpoint two interesting properties of the supertranspose:

$$\begin{aligned} (XY)^{ST} &= (-)^{|X||Y|} Y^{ST} X^{ST}, \\ (X^{ST})^{ST} &= \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}. \end{aligned}$$

Another important mapping is the supertranspose defined by

$$\text{sTr}(X) \stackrel{\text{df}}{=} \text{Tr}(A) - (-)^{|X|} \text{Tr}(D). \quad (20)$$

The following property of the supertrace hold:

$$\text{sTr}(XY) = (-)^{|X||Y|} \text{sTr}(Y) \text{sTr}(X).$$

Finally, the left and right scalar multiplication of  $\zeta \in \mathcal{C}\Lambda_N$  is defined as

$$\zeta X = \begin{pmatrix} \zeta A & \zeta B \\ (-)^{|\zeta|} \zeta C & (-)^{|\zeta|} \zeta D \end{pmatrix}, \quad (21)$$

$$X \zeta = \begin{pmatrix} A \zeta & (-)^{|\zeta|} B \zeta \\ C \zeta & (-)^{|\zeta|} D \zeta \end{pmatrix}. \quad (22)$$

Clearly, the grade of the Grassmann numbers make sense only for homogeneous elements but the linear character of supernumbers from Def. 3 extends its action for an arbitrary supernumber.

**Example.** We will not prove the listed properties of supermatrices [24] but let's see how the rules fit Def. 8 of a supermodule. Assume that  $\zeta \in A = \mathcal{C}\Lambda_N$  such that  $|\zeta| = 1$  and let  $X$  be an odd supermatrix. A simple example of such a matrix is when  $A = D = 0$  and  $B, C \in \mathbb{R}$ . From Eq. (21) we get

$$\zeta X = \begin{pmatrix} 0 & \zeta B \\ -\zeta C & 0 \end{pmatrix} = \begin{pmatrix} 0 & B \zeta \\ -C \zeta & 0 \end{pmatrix} = -X \zeta,$$

where the last equality comes from Eq. (22). The overall sign change coincides with Def. 8. Another non-trivial example is again for  $|\zeta| = 1$  but this time  $X$  is an even supermatrix realized by  $A = D = 0$  and  $B, C \in \mathcal{C}\Lambda_{N,1}$ . We can see that the following holds:

$$\zeta X = \begin{pmatrix} 0 & \zeta B \\ -\zeta C & 0 \end{pmatrix} = \begin{pmatrix} 0 & -B\zeta \\ C\zeta & 0 \end{pmatrix} = X\zeta.$$

Again, the overall sign agrees with the definition of a supercommutative module.

*Remark.* Note that Def. 8 comes before we introduced supermatrices. So strictly speaking the definition at that point is valid only for matrices from  $\text{End}(W^{p|q})$  with entries from  $\mathbb{R}$  or  $\mathbb{C}$ . But odd (grade-reversing) linear operators are examples of odd supermatrices as illustrated in the above example. More generally, a matrix  $L$  with entries in  $\mathbb{R}$  (or  $\mathbb{C}$ ) can be written as a sum of an even supermatrix  $L_e$  and an odd supermatrix  $L_o$

$$L = L_e + L_o = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \quad (23)$$

Now we can use the definition of the supertranspose introduced in Eq. (19) for supermatrices even for  $L \in W(p|q)$ .

$$L^{ST} = L_e^{ST} + L_o^{ST} = \begin{pmatrix} A^T & 0 \\ 0 & D^T \end{pmatrix} + \begin{pmatrix} 0 & -C^T \\ B^T & 0 \end{pmatrix}$$

due to the linearity of the supertranspose.

**Definition 9.** *The real orthosymplectic Lie superalgebra  $osp(p|q; \mathbb{R})$  is defined as*

$$osp(p|q; \mathbb{R}) = \{X \in gl(p|q; \mathbb{R}) | X^{ST}H + HX = 0\}.$$

*The matrix of a bilinear form reads*

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

*where  $H_1$  is a canonical symmetric matrix and  $H_2$  is a canonical skew-symmetric matrix.*

The algebra is a  $\mathbb{Z}_2$ -graded vector space where  $\dim H_1 = p$  and  $\dim H_2 = q$ . From the matrix representation of the bilinear form we see that the form is non-degenerate and the both subspaces are orthogonal with respect to it. We may rewrite the condition for a matrix  $X$  to be in  $osp(p|q; \mathbb{R})$  as

$$A^T H_1 + H_1 A = D^T H_2 + H_2 D = B^T H_1 - H_2 C = 0. \quad (24)$$

That's why the algebra is called orthosymplectic: the even subspace (in the sense of Def. 6) is a direct sum of two Lie algebras bearing the same name. The odd subspace does not form an algebra.

## II. SUPERQUBITS

The structures introduced in the previous section will help to understand all details of how superqubits are defined, their properties and relation to qubits.

**Definition 10.** The unitary orthosymplectic algebra  $uosp(1|2; \mathcal{C}\Lambda_N)$  is the Grassmannification of  $osp(1|2; \mathbb{R})$  defined as

$$uosp(1|2; \mathcal{C}\Lambda_N) = \{X \in osp(1|2; \mathbb{R}) \otimes \mathcal{C}\Lambda_N | X^\dagger = -X\},$$

where

$$X^\dagger \stackrel{\text{df}}{=} (X^{ST})^\#$$

is called the grade adjoint (superadjoint).

This is a crucial definition and we will spend some time analyzing its consequences. The algebra  $uosp(1|2; \mathcal{C}\Lambda_N)$  is not a Lie superalgebra in the sense of Def. 5. In  $osp(1|2; \mathbb{R}) \otimes \mathcal{C}\Lambda_N$  we recognize the structure of a supercommutative module from Def. 8 (we will use the left multiplication). The elements of  $osp(1|2; \mathbb{R}) \otimes \mathcal{C}\Lambda_N$  are therefore supermatrices.

*Remark.* Note that this definition of the grade adjoint will reduce to Def. 7 for elements of  $W(p|q)$ . The argument is identical to how we argued that the supertranspose can be used for ordinary matrices (see the remark before Def. 9).

We proceed with the following lemma.

**Lemma 11.** The elements of  $uosp(1|2; \mathcal{C}\Lambda_N)$  are even supermatrices.

*Proof.* It is the superadjoint constraint  $X^\dagger = -X$  that singles out even supermatrices. There are three even generators  $A_i$  of  $osp(1|2; \mathbb{R})$  and so they satisfy  $A_i^\dagger = -A_i$ . They are accompanied by two odd generators  $Q_i$ . It holds  $Q_i^\dagger = \epsilon^{ij} Q_j$  for the odd generators where  $\epsilon^{ij}$  is the anti-symmetric tensor ( $\epsilon^{12} = 1$ ) [25].

We will assume the existence of odd supermatrices and prove the statement by contradiction. Suppose  $X = \sum_i z_i \zeta_i A_i$  where  $z_i \in \mathbb{C}$  and  $|\zeta_i| = 1$ . Then from  $X^\dagger = -X$  and  $A_i^\dagger = -A_i$  it follows

$$\sum_i \bar{z}_i \zeta_i^\# = \sum_i z_i \zeta_i.$$

But this is impossible unless  $z_i = 0$  since for all odd supernumbers (and for any finite-dimensional Grassmann algebra) Eq. (7) dictates that  $(\zeta_i^\#)^\# = -\zeta_i$ .

Now suppose that  $X = \sum_i z_i \zeta_i Q_i$  where  $z_i \in \mathbb{C}$  and  $|\zeta_i| = 0$ . But  $X^\dagger = -X$  and that is clearly incompatible with  $Q_i^\dagger = \epsilon^{ij} Q_j$  unless  $z_i = 0$ .

One can easily verify that an arbitrary even element of the  $uosp(1|2; \mathcal{C}\Lambda_N)$  algebra is given by

$$X = \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3 + \zeta^\# Q_1 + \zeta Q_2, \quad (25)$$

and so indeed  $X^\dagger = -X$  is valid as long as  $\xi_i^\# = \xi_i \in \mathcal{C}\Lambda_{N,0}$  and  $\zeta \in \mathcal{C}\Lambda_{N,1}$  hold.  $\blacksquare$

As a consequence [21, 28] the  $uosp(1|2; \mathcal{C}\Lambda_N)$  algebra is an ordinary Lie algebra but for this to be true the Grassmann structure must be present. Before we proceed let's set  $N = 2$  for the order of the Grassmann algebra  $\mathcal{C}\Lambda_N$ . This is the lowest-dimensional non-trivial complex Grassmann algebra equipped with the grade involution. The case  $N = 1$  is impossible since at least two odd Grassmann generators are needed ( $\eta$  and its complex conjugate  $\eta^\#$ ). For  $N = 0$  the whole process is a mere complexification we are not interested in. Hence, following Def. 3 and the  $X^\dagger = -X$  condition we can further specify the coefficients of Eq. (25)  $\xi_i = a_i + b_i \eta \eta^\#$  and  $\zeta = p \eta$  where inevitably  $a_i, b_i, p \in \mathbb{R}$ .

*Remark.* The most general form of  $\zeta$  for  $N = 2$  is  $\zeta = p_1 \eta + p_2 \eta^\#$  for  $p_1, p_2 \in \mathbb{C}$ . By choosing  $\zeta = p \eta$  we further simplify the analysis.



Let's investigate what kind of Lie algebra we are dealing with. We can work with an explicit matrix representation (which happens to be fundamental) where  $A_j = i/2(0 \oplus \sigma_j)$  are embedded Pauli matrices  $\sigma_i$  and

$$Q_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We label  $X_1 = \eta^\# Q_1$  and  $X_2 = \eta Q_2$  and first find that  $[A_i, A_j] = -\epsilon_{ij}^k A_k$  which is the well known  $su(2)$  algebra. Thus it is a subalgebra of  $uosp(1|2; \mathcal{C}\Lambda_N)$  forming a connection to qubits we will elaborate more later on. Next we have

$$[A_1, X_1] = \frac{i}{2} X_2, \quad (26a)$$

$$[A_1, X_2] = \frac{i}{2} X_1, \quad (26b)$$

$$[A_2, X_1] = -\frac{1}{2} X_2, \quad (26c)$$

$$[A_2, X_2] = \frac{1}{2} X_1, \quad (26d)$$

$$[A_3, X_1] = \frac{i}{2} X_1, \quad (26e)$$

$$[A_3, X_2] = -\frac{i}{2} X_2 \quad (26f)$$

and finally the commutator closing the algebra:

$$[X_1, X_2] = \frac{i}{2} \eta \eta^\# A_3. \quad (27)$$

It is not immediately clear what kind of Lie algebra we obtained. An obvious candidate would be a semidirect sum of two subalgebras. There are indeed two subalgebras – the aforementioned  $su(2)$  and a subalgebra formed by the last two rows of Eq. (26) together with Eq. (27). This algebra is isomorphic to  $sl(2, \mathbb{R})$  (if we remap  $A_3$  to  $iA_3$ ). However, they ‘share’ a generator ( $A_3$ ) and so even if the rest of Eqs. (26) would favor the existence of a semidirect sum of two subalgebras there is none present. More information on how the structure constant and adjoint representation are defined for Grassmann-valued Lie algebras can be found on p. 217 of [21].

The main purpose for considering even supermatrices is that unlike the case of Lie superalgebras there exists an exponential map transforming the Grassmann number assisted Lie algebra into the corresponding Lie group [22, 28]. More precisely, there is an equivalent of the Zassenhaus formula for even supermatrices but not for orthosymplectic Lie superalgebras due to the presence of the anticommutator for odd elements of  $osp(p|q; \mathbb{R})$  [27] (similarly for the rest of Lie superalgebras).

**Definition 12.** *The  $UOSP(1|2)$  group is defined as*

$$UOSP(1|2) = \{Z = \exp X | X \in uosp(1|2; \mathcal{C}\Lambda_N)\},$$

where  $X$  comes from Eq. (25). An arbitrary group element can be written [28] as

$$Z = \exp [\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3] \exp [\zeta^\# Q_1 + \zeta Q_2]. \quad (28)$$

Recall that  $\xi_i = a_i + b_i \eta \eta^\#$  and  $\zeta = p\eta$ .

Note that the super adjoint condition on the algebraic level leads to the superunitary condition on the group level

$$Z^\dagger Z = ZZ^\dagger = 1. \quad (29)$$

We see how the pattern from ordinary Lie algebras/groups is closely followed but with different (actually more general) structures – grade adjoint and superunitary condition.

We will further simplify our analysis by taking an inspiration from the world of qubits. Qubits carry the fundamental representation of  $SU(2)$ . The space of qubits is not identified with the  $SU(2)$  group manifold (the  $S^3$  sphere) but rather with a coset space  $S^2 = SU(2)/U(1)$  (the Bloch sphere). The reason is that from the physical point of view there is a redundancy in the form of an overall phase generated by  $U(1)$ . This is all well known but let's make it explicit to compare it with what follows for superqubits. If we exponentiate an arbitrary element of the  $su(2)$  Lie algebra we obtain

$$V = \exp \sum_i a_i A_i = \cos \frac{m\theta}{2} \mathbb{1} + \frac{i}{m} \sin \frac{m\theta}{2} (a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3),$$

where  $m = \sum_i a_i^2$ . The explicit transition from  $SU(2)$  to  $SU(2)/U(1)$  is achieved by setting  $m = 1$ .

For the  $SU(2)$  part of  $UOSP(1|2)$  the exponentiation goes through in exactly the same way. The only difference is that the parameter  $m = \sum_i a_i^2 + 2a_i b_i \eta \eta^\#$  is even Grassmann. We have essentially two options how to proceed. We can either define an even Grassmann version of  $U(1)$

$$U(0|1) = \{x \in \mathcal{C}\Lambda_{2,0}; xx^\# = 1\}$$

and introduce a projective space  $S^{2|2} = UOSP(1|2)/U(0|1)$  known as the supersphere [29–32]. This equals to setting  $m = \sum_i a_i^2 + 2a_i b_i \eta \eta^\# = 1$  and so

$$\sum_i a_i^2 = 1 \text{ and } \sum_i a_i b_i = 0.$$

The simplification we will make is to take the trivial solution where  $b_i = 0$  for all  $i$ . So instead of subtracting  $U(0|1)$  from  $UOSP(1|2)$  to get  $S^{2|2}$  we can subtract  $U(1)$ . Let's finally define a superqubit.

**Definition 13** ([14, 29–32, 35]). *A superqubit is a carrier of the fundamental representation of the group  $UOSP(1|2)$  introduced in Def. (12) where we set  $N = 2$ ,  $\sum_i a_i^2 = 1$ ,  $b_i = 0$  and  $p \in \mathbb{R}$ . The space of superqubits is a coset space  $S^{2|2} = UOSP(1|2)/U(1)$  that can be called the super Bloch sphere.*

We gain a further insight into the superqubit space by performing the exponentiation in Eq. (28). We find

$$Z(2p\eta, \alpha, \beta) = U(\alpha, \beta)S(2p\eta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\beta^\# \\ 0 & \beta & \alpha^\# \end{pmatrix} \begin{pmatrix} 1 + p^2 \eta \eta^\# & -p\eta & p\eta^\# \\ -p\eta^\# & 1 - \frac{p^2}{2} \eta \eta^\# & 0 \\ -p\eta & 0 & 1 - \frac{p^2}{2} \eta \eta^\# \end{pmatrix}, \quad (30)$$

where  $\alpha = \cos \vartheta, \beta = e^{i\phi} \sin \vartheta$  is the usual reparametrization of the Bloch sphere. Note that due to the choice  $b_i = 0$  we made above  $\alpha, \beta \in \mathbb{C}$  and so  $\alpha^\# \equiv \bar{\alpha}, \beta^\# \equiv \bar{\beta}$ . In this matrix representation the basis states read

$$|\bullet\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (31)$$

The basis states  $|0\rangle$  and  $|1\rangle$  are even (bosonic) states. The basis state  $|\bullet\rangle$  has a distinguished notation introduced in [14] to stress that the basis state is odd (fermionic). Why exactly do we call them bosonic and fermionic states and are the bosonic bases the qubit basis states (as the notation  $|0\rangle$  and  $|1\rangle$  already suggests)? The answers lie in the Jordan-Schwinger (oscillator) representation of  $su(2)$  and  $osp(1|2)$ . The  $su(2)$  algebra is represented by

$$A_1 = \frac{i}{2}(b_1^\dagger b_2 + b_2^\dagger b_1), \quad (32a)$$

$$A_2 = \frac{1}{2}(b_1^\dagger b_2 - b_2^\dagger b_1), \quad (32b)$$

$$A_3 = \frac{i}{2}(b_1^\dagger b_1 - b_2^\dagger b_2), \quad (32c)$$

where  $b_i, b_i^\dagger$  are operators satisfying the canonical commutation relation and  $A_i^\dagger = A_i^\dagger = -A_i$ . The even part of the  $osp(1|2)$  algebra can be transformed to two ordinary Lie algebras that are its real forms: either the  $su(2)$  algebra as its compact bosonic subalgebra or the  $su(1,1)$  algebra that is non-compact. We are interested in the former. Hence we are dealing with the following Lie superalgebra

$$[A_i, A_j] = -\epsilon_{ij}^k A_k, \quad (33a)$$

$$[A_i, Q_j] = \frac{i}{2}(\sigma_i)_{kj} Q_k, \quad (33b)$$

$$\{Q_1, Q_2\} = \frac{i}{2} A_3, \quad (33c)$$

$$\{Q_1, Q_1\} = \frac{1}{2}(A_2 - iA_1), \quad (33d)$$

$$\{Q_2, Q_2\} = \frac{1}{2}(A_2 + iA_1), \quad (33e)$$

where the curly brackets denote the anticommutator. We complement Eqs. (32) with

$$Q_1 = \frac{1}{2}(f^\dagger b_2 + f b_1^\dagger), \quad (34a)$$

$$Q_2 = \frac{1}{2}(f b_2^\dagger - f^\dagger b_1), \quad (34b)$$

where  $f, f^\dagger$  are operators satisfying the canonical anticommutation relation  $\{f, f^\dagger\} = 1$ . One can verify that the operator representation Eqs. (32) and (34) satisfies Eqs. (33) but not  $Q_i^\dagger = \epsilon^{ij} Q_j$ . Instead, it satisfies  $Q_i^\dagger = -\epsilon^{ij} Q_j$ .

*Remark.* This problem can be avoided if we allow the two adjoints to coexist as in [36]. Recall that the adjoint is an automorphism of ordinary fermion algebra  $\{f, f^\dagger\} = 1$ . Similarly, the superadjoint is an automorphism of our superfermions

$$\{f, f^\dagger\} = f f^\dagger + f^\dagger f \xrightarrow{\dagger} f f^\dagger + f^\dagger f = \{f, f^\dagger\} = 1$$

using Eqs. (12) and (13). Finally, there seems to be no inconsistency to let both adjoints act on the same structure. Something similar is already happening for complex Grassmann algebras where both types of automorphisms from Def. 4 can be used at the same time. For

$$\{f, f^\dagger\} \xrightarrow{\dagger} (f^\dagger)^\dagger f^\dagger + f^\dagger (f^\dagger)^\dagger = 1$$

to hold we have to assume  $(f^\dagger)^\dagger = f$  and  $f^\dagger = -f^\dagger$ . This looks unusual but that is precisely the result from [36]. Now we can again define

$$Q_1 = \frac{1}{2}(f^\dagger b_2 + f b_1^\dagger), \quad (35a)$$

$$Q_2 = \frac{1}{2}(f b_2^\dagger - f^\dagger b_1). \quad (35b)$$

The supercommutators from Eqs. (33) stay intact using the new definition and  $Q_i^\dagger = \epsilon^{ij} Q_j$  will be indeed satisfied.

*Remark.* The second line of Eq. (33) is formally similar to Eqs. (26). Recall, however, that contrary to the algebra  $uosp(1|2; \mathcal{CL}_N)$  introduced in Def. 10, Eqs. (33) do form a Lie superalgebra.

The purpose of boson/fermion creation and annihilation operators is twofold. They are building blocks of algebra representations but they also pave the way to the proper definition of a Hilbert space. Let's introduce a 'vacuum' state  $|vac\rangle$  defined such that  $b_1 |vac\rangle = b_2 |vac\rangle = f |vac\rangle = 0$  is satisfied. In the 'single-particle' sector we get

$$|\bullet\rangle = f^\dagger |vac\rangle, \quad |0\rangle = b_1^\dagger |vac\rangle, \quad |1\rangle = b_2^\dagger |vac\rangle. \quad (36)$$

These states are precisely the bases from Eq. (31) and so we have justified the name bosonic for even states and fermionic for odd states. Let's call this set of states the canonical basis. It remains to be argued why  $\{|0\rangle, |1\rangle\}$  is called a qubit basis. Naively, it seems sufficient to say that the states are eigenstates of  $A_3$  that corresponds to  $\sigma_3$  for qubits. But one has to ask what happens for a system of two or more bosons and whether their exchange statistics conform with how a system of qubits behave when two or more qubits are swapped. The answer is that unlike a system of fermions we can indeed associate  $N$  distinguishable bosons with  $N$  qubits. Note that in our case the parameter that distinguishes the two bosons  $b_i, (b_i^\dagger)$  is the index  $i$ .

The space spanned by vectors from Eq. (36) is a  $\mathbb{Z}_2$ -graded Hilbert space. But once we start constructing even particle sectors we run into troubles. It turns out that this is an example of a grade star representation where by taking a tensor product of two such representations we obtain a vector space that is not a Hilbert space (see the discussion after Def. 7). To see this let's have a lemma.

**Lemma 14.** *The grade adjoint of a tensor product of two linear operators from  $\text{End}(W^{p|q})$  satisfy*

$$\dagger : f_1 f_2 \mapsto f_1^\dagger f_2^\dagger. \quad (37)$$

We postpone the proof but let's present an argument in favor of the statement's validity. From Eq. (13) we know that  $(f^\dagger)^\dagger = -f$  since  $f$  is an odd operator. The product of three fermion operators is odd as well and so

$$((f_1 f_2 f_3)^\dagger)^\dagger = -f_1 f_2 f_3.$$

Clearly, the proposed action of the grade adjoint in Eq. (37) is compatible with the above equation

$$((f_1 f_2 f_3)^\dagger)^\dagger = (f_1^\dagger (f_2 f_3)^\dagger)^\dagger = (f_1^\dagger f_2^\dagger f_3^\dagger)^\dagger = -f_1 f_2 f_3.$$

Another version of Eq. (37) is  $\dagger : f_1^\dagger f_2^\dagger \mapsto f_1 f_2$  and so by using  $|\bullet\rangle_1 |\bullet\rangle_2 = f_1^\dagger f_2^\dagger |vac\rangle$  and its grade adjoint  $\langle\bullet|_1 \langle\bullet|_2 = \langle vac| f_1 f_2$  we obtain the norm of this state to be

$$\langle\bullet|_1 \langle\bullet|_2 |\bullet\rangle_1 |\bullet\rangle_2 = (-)^{|\bullet||\bullet|} \langle\bullet|\bullet\rangle_1 \langle\bullet|\bullet\rangle_2 = -\langle\bullet|\bullet\rangle_1 \langle\bullet|\bullet\rangle_2 = -1. \quad (38)$$

*Remark.* If we used the properties of the ordinary adjoint (Eq. (15)) we would find that

$$((f_1 f_2 f_3)^\dagger)^\dagger = f_1 f_2 f_3.$$

The two mappings are indeed different in many aspects.

Let's present the real proof in the framework of the group we are interested in –  $UOSP(1|2)$ . First, we observe that the matrix  $U(\alpha, \beta)$  is an element of the  $SU(2)$  subgroup as a consequence of the  $su(2)$  subalgebra of  $uosp(1|2, \mathcal{C}\Lambda_2)$ . The second matrix is more interesting. Here come two simple lemmas studying its properties.

**Lemma 15.** *The matrix  $S(2p\eta)$  is an element of an Abelian group isomorphic to  $(\mathbb{R}, +)$  with the group operation being addition.*

*Proof.* One can easily verify the group axioms:

- (i)  $S(2p\eta)S(2q\eta) = S(2(p+q)\eta)$ .
- (ii) There exists an identity  $S(0) = I$ .
- (iii) There exists an inverse element  $S^{-1}(2p\eta) = S(-2p\eta) \equiv S^\dagger(2p\eta)$ .

■

The group  $(\mathbb{R}, +)$  is non-compact – an issue whose solution we will offer later. Next, we show that the two matrices  $U$  and  $S$  ‘essentially commute’.

**Lemma 16.**  $U(\alpha, \beta)S(2p\tilde{\eta}) = S(2p\eta)U(\alpha, \beta)$  where  $\tilde{\eta} = \alpha\eta - \beta\eta^\#$ .

*Proof.* The claim can be proved directly by matrix multiplication but it is easier (and sufficient) to show that

$$U(\alpha, \beta)S(2p\tilde{\eta})|m\rangle = S(2p\eta)U(\alpha, \beta)|m\rangle,$$

where  $m = \bullet, 0, 1$ .

■

As follows from Eq. (30), a general form of a pure superqubit is

$$|\psi\rangle = S(2p\eta)U(\alpha, \beta)|0\rangle = \left(1 - \frac{p^2}{2}\eta\eta^\#\right)(\alpha|0\rangle + \beta|1\rangle) + p(-\alpha\eta + \beta\eta^\#)|\bullet\rangle. \quad (39)$$

Using Def. 7 we find the corresponding bra vector

$$\langle\psi| = \left(1 - \frac{p^2}{2}\eta\eta^\#\right)(\alpha^\# \langle 0| + \beta^\# \langle 1|) + p(\alpha^\# \eta^\# + \beta^\# \eta) \langle \bullet|. \quad (40)$$

In Lemma 11 we showed that the elements of the  $uosp(1|2; \mathcal{C}\Lambda_2)$  algebra are even supermatrices. One of the consequences is that superqubits are even vectors and so  $|\psi\rangle_1 |\psi\rangle_2 = |\psi\rangle_2 |\psi\rangle_1$ . This can be readily verified from Eq. (39) by a direct calculation.

*Remark.* Def. 13 coincides with the original definition of a superqubit [14]. After the change of variables that happens to be almost identical to the transformation of Grassmann variables in Lemma 16

$$-\alpha\eta + \beta\eta^\# \mapsto \eta, \quad -\alpha^\#\eta^\# - \beta^\#\eta \mapsto \eta^\#,$$

the state (39) becomes

$$|\psi\rangle = \left(1 - \frac{p^2}{2}\eta\eta^\#\right)(\alpha|0\rangle + \beta|1\rangle) + p\eta|\bullet\rangle. \quad (41)$$

**Example.** We can illustrate the subtleties of mundane calculations on (super)density matrices. Assuming  $\alpha = 1, \beta = 0$  for simplicity, Eq. (39) leads to the superdensity matrix of the form

$$\varrho = |\psi\rangle\langle\psi| = \begin{pmatrix} -p^2\eta\eta^\# & -p\eta \\ p\eta^\# & 1 - p^2\eta\eta^\# \end{pmatrix}. \quad (42)$$

The supertrace Eq. (20) can be rewritten as  $\text{sTr}(X) = \sum_i (-)^{|i|} x_{ii}$  where  $x_{ii} = \langle i | X | i \rangle$  giving us  $\text{sTr}(\varrho) = 1$ . The supertranspose is more tricky and we should be guided by the properties of the superadjoint:

$$\begin{aligned} \varrho &= -p^2\eta\eta^\# |\bullet\rangle\langle\bullet| + p\eta^\# |0\rangle\langle\bullet| - p\eta |\bullet\rangle\langle 0| + (1 - p^2\eta\eta^\#) |0\rangle\langle 0| \\ &\xrightarrow{\dagger} -p^2\eta\eta^\# |\bullet\rangle\langle\bullet| - p\eta |\bullet\rangle\langle 0| + p\eta^\# |0\rangle\langle\bullet| + (1 - p^2\eta\eta^\#) |0\rangle\langle 0| = \varrho^\dagger. \end{aligned}$$

Hence  $\varrho = \varrho^\dagger$  and the superdensity matrix is super Hermitian. If we attempt to do the same calculation with the matrix of  $\varrho$  (Eq. (42)) then following Eq. (19) we indeed get the correct result:  $\varrho^\dagger = (\varrho^\#)^{ST} = \varrho$ . However, when we merely permute the basis and write

$$\varrho = \begin{pmatrix} 1 - p^2\eta\eta^\# & p\eta^\# \\ -p\eta & -p^2\eta\eta^\# \end{pmatrix} \quad (43)$$

we found that  $(\varrho^\#)^{ST} \neq \varrho$ . The reason is that the supertranspose is heavily basis-dependent similarly to the ordinary transpose. When we change the basis (even by a mere permutation of basis vectors) we have to alter the definition of the supertranspose as well to get the right answer.

Let's pause for a moment and summarize some of the rules we use here. There is an interesting question whether odd states (such as  $|\bullet\rangle = f^\dagger |vac\rangle$ ) commute or anticommute with odd Grassmann numbers. They belong to two different structures that are nevertheless closely related. Grassmann numbers obey Grassmann algebra and  $f(f^\dagger)$  belong to the algebra of fermions. The common feature is that two elements belonging to the same algebra type anticommute:

$$\eta_1\eta_2 = -\eta_2\eta_1, \quad f_1^\dagger f_2^\dagger = -f_2^\dagger f_1^\dagger$$

( $\eta_i$  are odd Grassmann numbers). The answer to the question of what happens when an odd Grassmann number passes through an odd state lies in the definition of a supercommutative module Def. 8. In the example following Def. 8 we showed how the left and right multiplication by odd Grassmann numbers (Eqs. (21) and (22)) satisfy the supermodule definition. Let's revisit the example. Assume  $\zeta \in \mathcal{CA}_N$  to be odd and let  $X$  be an example of an odd supermatrix ( $A = D = 0$  and  $B, C \in \mathbb{R}$ ). Following Eq. (21) we get

$$\zeta X = \begin{pmatrix} 0 & \zeta B \\ -\zeta C & 0 \end{pmatrix} = \begin{pmatrix} 0 & B\zeta \\ -C\zeta & 0 \end{pmatrix} = -X\zeta.$$

Rewritten in the basis Eq. (31) we have

$$\begin{aligned} \zeta X &= -\zeta c_0 |0\rangle\langle\bullet| - \zeta c_1 |1\rangle\langle\bullet| + \zeta b_0 |\bullet\rangle\langle 0| + \zeta b_1 |\bullet\rangle\langle 1| \\ &= -c_0 |0\rangle\langle\bullet| \zeta - c_1 |1\rangle\langle\bullet| \zeta + b_0 |\bullet\rangle\langle 0| \zeta + b_1 |\bullet\rangle\langle 1| \zeta = -X\zeta, \end{aligned}$$

where  $c_i, b_i \in \mathbb{R}$  forming  $C$  and  $B$ , respectively. We immediately see that all odd Grassmann numbers commute with odd states. Therefore all supernumbers (even and odd) commute with all states (even and odd). Here is a comprehensive list of the rules we use:

- (i)  $\zeta |\bullet\rangle = |\bullet\rangle \zeta$

- (ii)  $\zeta \langle \bullet | = \langle \bullet | \zeta$
- (iii)  $\ddagger : |\bullet\rangle \mapsto -\langle \bullet |$
- (iv)  $\ddagger : \zeta |\bullet\rangle \mapsto -\zeta^\# \langle \bullet |$
- (v)  $\ddagger : \zeta \langle \bullet | \mapsto \zeta^\# |\bullet\rangle$

valid for an arbitrary  $\zeta \in \mathcal{C}\Lambda_N$ .

*Remark.* Note that these rules are different compared to [14] where the position of a Grassmann number with respect to a ket/bra has to be carefully tracked. The current convention is arguably more friendly for calculation purposes. Both conventions, however, are consistent and the difference boils down to two types of the so-called Grassmann envelope of a vector space [21]. The Grassmann envelope of the  $\mathbb{Z}_2$ -graded vector space  $W^{p|q}$  corresponds to the structure we call here a supercommutative module (Def. 8). The other convention is related to the following modification of how the left and right multiplications are related in Eq. (16):  $av = -(-)^{|a||v|}va$ .

Having established all the necessary rules for calculating with Grassmann numbers we can settle our debts and prove Lemma 14.

*Proof of Lemma 14.* Superqubits (Eq. (39)) are normalized to one. Hence

$$1 = \langle \psi | \psi \rangle_1 \langle \psi | \psi \rangle_2 = \langle \psi |_1 \langle \psi |_2 | \psi \rangle_1 | \psi \rangle_2 \quad (44)$$

The grade adjoint must satisfy  $\ddagger : |\psi\rangle_1 |\psi\rangle_2 \mapsto \langle \psi |_1 \langle \psi |_2$ . Following Eq. (40) we write

$$\begin{aligned} \langle \psi |_1 \langle \psi |_2 &= \left(1 - \frac{p^2}{2} \eta_1 \eta_1^\# \right) \left(1 - \frac{p^2}{2} \eta_2 \eta_2^\# \right) \langle 0 |_1 \langle 0 |_2 + \left(1 - \frac{p^2}{2} \eta_1 \eta_1^\# \right) p \eta_2^\# \langle 0 |_1 \langle \bullet |_2 \\ &\quad + \left(1 - \frac{p^2}{2} \eta_2 \eta_2^\# \right) p \eta_1^\# \langle \bullet |_1 \langle 0 |_2 + p^2 \eta_1^\# \eta_2^\# \langle \bullet |_1 \langle \bullet |_2, \end{aligned} \quad (45)$$

where we assume  $\alpha = 1, \beta = 0$  and  $p_1 = p_2 = p$  for simplicity. Clearly, the grade adjoint satisfies  $\ddagger : |\psi\rangle_1 |\psi\rangle_2 \mapsto \langle \psi |_1 \langle \psi |_2$  only if  $\ddagger : f_1^\ddagger f_2^\ddagger \mapsto f_1 f_2$  holds. In detail, the double-bullet component of  $|\psi\rangle_1 |\psi\rangle_2$  transforms as

$$p^2 \eta_1 \eta_2 |\bullet\rangle_1 |\bullet\rangle_2 \equiv p^2 \eta_1 \eta_2 f_1^\ddagger f_2^\ddagger |vac\rangle \xrightarrow{\ddagger} p^2 \eta_1^\# \eta_2^\# \langle vac |_1 f_1 f_2 \equiv p^2 \eta_1^\# \eta_2^\# \langle \bullet |_1 \langle \bullet |_2.$$

This is precisely the last summand of Eq. (45). ■

Does it mean that after so much work we don't even have a proper Hilbert space? Fortunately, the answer is no and there are two reasons for it. First, looking at Eq. (30) we notice something unusual. The  $UOSP(1|2)$  does not act transitively and so the superqubit space is not a homogeneous space. There is no unitary  $Z \in UOSP(1|2)$  that would take us from a subspace spanned by  $\{|0\rangle, |1\rangle\}$  to the subspace spanned by  $|\bullet\rangle$ . In principle, we could define even superqubits like Eq. (39) and odd superqubits  $S(2p\eta)U(\alpha, \beta) |\bullet\rangle$  that would not be equivalent. However, a tensor product of two odd superqubits would suffer from the same problem as the state  $|\bullet\bullet\rangle_{12}$  – its norm would be negative (see Lemma 14). So we will consider only even superqubits and call them simply superqubits (single or multipartite).

The second key aspect is the transition from Lie superalgebras to Grassmann-valued Lie algebras we underwent in Def. 10. The constraint on even operators is nothing else than a super version of antihermiticity. We can trivially rewrite the constraint as  $X^\ddagger G + GX = 0$  where

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (46)$$

is a matrix representing a non-degenerate positive semidefinite form that again appears right after Def. 12:  $Z^\dagger G Z = G$ . We implicitly used this metric when we normalized the superqubit in Eq. (39). This choice is important yet from another reason than positivity. The  $UOSP(1|2)$  group acts as an isometry group on a vector space equipped with the inner product induced by  $G$ .

**Definition 17.** Let  $\mathcal{S}$  be non-degenerate form

$$\mathcal{S} : V \times V \mapsto \mathcal{C}\Lambda_{N,0},$$

where  $V$  is the carrier space of the  $UOSP(1|2)$  group. We ask for the following properties to be satisfied:

- (i)  $\mathcal{S}(v, v) \geq 0$  (positive semidefinite)
- (ii)  $\mathcal{S}(u, v) = \mathcal{S}(v, u)^\#$  (super Hermitian)
- (iii)  $\mathcal{S}(\alpha u, v) = \bar{\alpha} \mathcal{S}(u, v)$  (sesquilinear)
- (iv)  $\mathcal{S}(u, v)_\mathbb{C} = \mathcal{S}(u_\mathbb{C}, v_\mathbb{C})$  (consistent),

where  $u, v \in V, \alpha \in \mathbb{C}$  and the index  $\mathbb{C}$  in case (iv) denotes the complex (non-Grassmann) part of a supernumber or any other (super)object.

The definition has interesting consequences we will discuss in detail now.

**Proposition 18.** Let  $\langle | \rangle$  be an inner product induced by  $G(e_i, e_j) = g_{ij}$  (Eq. (46)) defined as

$$\langle u | v \rangle = u^{i\#} v^j g_{ij}, \quad (47)$$

where  $u = u^i e_i, v = v^j e_j \in V$  are superqubits,  $u^i, v^j \in \mathcal{C}\Lambda_2$  and  $e_i$  form an orthogonal basis Eq. (36) ( $i, j = \{0, 1, \bullet\}$ ). Then the inner product satisfies the properties listed in Def. 17.

*Remark.* First of all, it is clear that  $\langle Zu | Zv \rangle = \langle u | Z^\dagger Z v \rangle = \langle u | v \rangle$  for all  $u, v \in V$  and  $Z \in UOSP(1|2)$  since  $V$  is the carrier space of the group. But recall that the carrier space is not an ordinary vector space – it is a supercommutative module due to the presence of Grassmann variables. In ordinary complex vector spaces there wouldn't be a reason to prove anything.  $G$  would be a metric preserved by  $SU(3)$  and the axioms from Def. 17 would become trivial or satisfied by definition.

*Proof.* (i) We know that this is true already from Eq. (39). But suppose we are given the most general superqubit  $v = v^0 e_0 + v^1 e_1 + v^\bullet e_\bullet$  where  $v^0, v^1 \in \mathcal{C}\Lambda_{2,0}$  and  $v^\bullet \in \mathcal{C}\Lambda_{2,1}$ . We find that the requirement  $\langle v | v \rangle = 1$  (to normalize it to one as in quantum mechanics) equals to

$$\tilde{v} = (v^{k\#} v_k)^{-1/2} v^i e_i.$$

But Grassmann numbers can be inverted only if the non-Grassmann part is nonzero. If we assume so  $\tilde{v}$  becomes the superqubit Eq. (41) if we substitute  $p \mapsto -p$ . (ii) This follows from the definition of the grade involution Def. 4

$$(\langle v | u \rangle)^\# = (v^{i\#} u_i)^\# = (-)^{|v^i|} v^i u_i^\# = u_i^\# v^i = \langle u | v \rangle.$$

The third inequality is valid for both  $v^i$  even and odd. If  $v^i$  is even then  $u_i^\#$  is even as well (recall that  $g_{ij}$  is diagonal),  $(-)^{|v^i|} = 1$  and they commute. If  $v^i$  is odd then  $u_i^\#$  is odd as well,  $(-)^{|v^i|} = -1$  and they anticommute. This cancels the minus sign and so we always



obtain  $u_i^\# v^i$ . (iii) This immediately follows from the definition of the inner product and the fact that  $\#$  acts as ordinary complex conjugation for  $\alpha \in \mathbb{C}$ . (iv)  $v^0, v^1$  are Grassmann even and so they also contain purely complex components. The same holds for their product and hence  $\langle u|v \rangle_{\mathbb{C}} = (u^0 v_0 + u^1 v_1)_{\mathbb{C}} = \bar{u}_{\mathbb{C}}^0 v_{0,\mathbb{C}} + \bar{u}_{\mathbb{C}}^1 v_{1,\mathbb{C}}$ . On the other hand, we immediately get  $\langle u_{\mathbb{C}}|v_{\mathbb{C}} \rangle = \bar{u}_{\mathbb{C}}^0 v_{0,\mathbb{C}} + \bar{u}_{\mathbb{C}}^1 v_{1,\mathbb{C}}$ . ■

The consistency condition (case (iv)) has an impact from the physical point of view:

**Corollary 19.** *Quantum theory based on superqubits is not a modification of quantum mechanics but rather its extension to the specific supersymmetric domain.*

Indeed, if we threw away the Grassmann part of the superqubit state (by setting  $p = 0$  in Eq. (39)) the state reduces to an ordinary qubit requiring no further action) and all the super structures we introduced would become the familiar constructions from quantum theory. The kind of supersymmetry we study here extends quantum theory to the supersymmetric domain but does not alter its current form – there is absolutely no experimental evidence for quantum theory to be invalid. To complete the ‘proof’ of the corollary one last thing remains to be analyzed – the definition of measurement probability.

Before we delve into the discussions of how to interpret Grassmann variables as probabilities we have to close the question of the existence of a Hilbert space for multipartite states. Def. 17 only explicitly talks about single superqubits and the question can only be fully resolved after higher Lie superalgebras and groups have been studied. But we can say something already now. If we take a tensor product of  $k$  superqubits it is clear that they will live in a subspace superunitarily connected with the usual  $k$ -qubit basis of dimension  $2^k$ . So, for instance, the state  $|\bullet\bullet\rangle_{12}$  that causes so much trouble is not a valid two-superqubit state – there is no superunitary  $Z_1 \otimes Z_2$  that would transform any state from the even two-superqubit subspace to  $|\bullet\bullet\rangle_{12}$ . Another example would be the state  $|\psi\rangle_{12} = |01\rangle - |10\rangle + |\bullet\bullet\rangle$ . Note that its norm equals one but this state does not belong to the Hilbert space introduced above.

We continue to construct our supersymmetric extension of quantum mechanics.

**Definition 20.** *Let the Grassmann-valued transition probability function between two superqubits  $\varphi$  and  $\psi$  be defined as*

$$p_{\mathcal{G}}(\varphi, \psi) = \langle \varphi | \psi \rangle (\langle \varphi | \psi \rangle)^\#. \quad (48)$$

The rationale behind the definition is clear. For ordinary (non-Grassmann) states we want to recover Born’s rule. We first note that  $p_{\mathcal{G}}(\psi, \psi) = 1$  for all  $\psi$ . The question we face now is how to interpret Grassmann-valued transition probability functions. Is there something special about the Grassmann numbers that are obtained by means of Eq. (48) for any two superqubits  $\varphi, \psi$ ? It turns out that this is the case. Such Grassmann numbers are not only always even (follows from Prop. 18 (ii)) but also satisfy the ‘reality condition’:

**Definition 21.** *An even Grassmann number  $\zeta \in \mathcal{C}\Lambda_{N,0}$  will be called real if  $\zeta^\# = \zeta$ .*

As a small detour, due to the reality condition definition we can actually gain some fresh insight into the origin of the two types of automorphisms from Def. 4.

**Lemma 22.** *Let  $\text{conj} : \Lambda_N \mapsto \Lambda_N$  be an antilinear map. For an arbitrary  $\zeta \in \Lambda_N$  we define the reality condition on  $\zeta$  to be*

$$\zeta \text{ conj}(\zeta) = \text{conj}(\zeta \text{ conj}(\zeta)). \quad (49)$$

*Then there are at least two types of conjugations satisfying the reality condition.*

*Proof.* First suppose that the map is an antiautomorphism. The left side of Eq. (49) becomes

$$\text{conj}(\zeta \text{conj}(\zeta)) = \text{conj}(\text{conj}(\zeta)) \text{conj}(\zeta). \quad (50)$$

For it to be equal to the RHS of Eq. (49) it requires the map  $\text{conj}$  to be an involution:

$$\text{conj}(\text{conj}(\zeta)) = \zeta. \quad (51)$$

The map is then the usual complex conjugation defined for Grassmann variables in quantum field theory of fermions [44] (the star map from Def. 4)

$$(\zeta^*)^* = \zeta.$$

The second option is an order preserving type of conjugation

$$\text{conj}(\zeta \text{conj}(\zeta)) = \text{conj}(\zeta) \text{conj}(\text{conj}(\zeta)). \quad (52)$$

In order to satisfy the RHS of Eq. (49) it must hold that

$$\text{conj}(\text{conj}(\zeta)) = -\zeta. \quad (53)$$

So this kind of conjugation is precisely the hash map also introduced in Def. 4 and used throughout this work

$$(\zeta^\#)^\# = -\zeta.$$

For  $\zeta \in \mathbb{C}$  both maps become ordinary complex conjugation and Eq. (49) is trivially satisfied. ■

*Remark.* It might be interesting to show how many more mappings there are for Grassmann variables that satisfy the reality condition.

Let's go back to the interpretation of Grassmann variables. We are not the first ones to ask about their meaning [37]. The pioneering work in this direction had been done by A. Rogers and others in the 80's [38]. The motivation there was then the burgeoning field of superanalysis on supermanifolds [39–41] as a response to the discovery of supersymmetric theories in physics. This is a branch of mathematics on its own indirectly related to the topic of this work. We will just define the Rogers prescription of how to extract ordinary numbers from Grassmann numbers and see if it can be of use for us. Of course, the reason why Grassmann numbers cannot be used directly is that they cannot be ordered in the first place. But there is another, closely related, reason. The outputs of measurement devices are real numbers as well as the outcomes probabilities and we would like to have an elegant prescription à la quantum mechanics.

**Definition 23** (The Rogers norm [38]). *Let  $\zeta \in \mathcal{CA}_N$  be an arbitrary supernumber whose general form was introduced in Def. 3. The Rogers norm of  $\zeta$  is defined as*

$$|\zeta|_{R_1} \stackrel{\text{df}}{=} |z_0| + \sum_{k=1}^N \sum_{m=1}^{\binom{N}{k}} |z^{(m)}|. \quad (54)$$

Spaces equipped with the Rogers norm teem with many interesting properties we will not discuss here [38]. From a broader point of view it is probably the most straightforward way of extracting real numbers from Grassmann numbers – one simply looks at the accompanying coefficients. So even though our situation is different (we want to interpret even Grassmann-valued probabilities), the most natural way to do so will be similar.

If we applied the Rogers norm directly to  $p_{\mathcal{G}}(\varphi(q), \psi(p)) = 1 - (p - q)^2 \eta \eta^{\#}$  (suppose  $\alpha = 1, \beta = 0$  for both states) calculated according to Def. 20 for superqubits (Eq. (39)) we would face a problem:

$$|p_{\mathcal{G}}(\varphi(q), \psi(p))|_{R_1} = 1 + (p - q)^2. \quad (55)$$

The Rogers norm does not respect the order of Grassmann variables and as a consequence we would get real number impossible to interpret as probabilities. Notice that we get the same result if we swap the Grassmann generators  $|1 + (p - q)^2 \eta^{\#} \eta|_{R_1} = 1 + (p - q)^2$ . So a slight modification of the Rogers norm has been proposed in [15] where the two main differences are: (i) the modified Rogers norm respects the order of Grassmann generators that must be fixed during the whole calculation and (ii) the modified Rogers norm transforms even Grassmann-valued probability functions. This enables us to recast the calculation of the modified Rogers norm into a form familiar from the path integral formulation of QFT – a Berezin (or Grassmann) integral [44]. This is another known way of how to extract real numbers from Grassmann variables (for other possibilities cf. [42, 43]).

**Definition 24** (The modified Rogers norm [15]). *Let  $\tau \in \mathcal{C}\Lambda_{N,0}$  be an even Grassmann number. The modified Rogers norm is defined*

$$|\tau|_R \stackrel{\text{df}}{=} \int d^{2N} \eta \prod_{i=1}^{N/2} e^{\eta_i \eta_i^{\#}} \tau, \quad (56)$$

where  $d^{2N} \eta \stackrel{\text{df}}{=} \prod_{i=1}^{N/2} d\eta_i d\eta_i^{\#}$  and  $\int d^{2N} \eta \prod_i \exp(\eta_i \eta_i^{\#}) = 1$ .

Recall that we consider Grassmann algebras where  $N = 2k$  for  $1 \leq k < \infty$ .

*Remark.* Literature on fermionic path integral is divided regarding the definition of Grassmann integral [44–47]. This is due to how a complex Grassmann algebra is understood. Let's elaborate on this issue a bit more first using the star involution from Def. 4. Usually, one starts with a real Grassmann algebra of order  $2k$  generated by  $\{\theta_i\}_{i=1}^{2k}$  and define the single-variable Grassmann integral

$$\int d\theta_k \theta_j \stackrel{\text{df}}{=} \delta_{jk}.$$

The algebra can be complexified

$$\begin{aligned} \eta_j &= \frac{1}{\sqrt{2}}(\theta_j + i\theta_{j+1}) \\ \eta_j^* &= \frac{1}{\sqrt{2}}(\theta_j - i\theta_{j+1}), \end{aligned}$$

where  $j = 1 \dots k$ . Hence [45, 47]

$$\begin{aligned} d\eta_j &= \frac{1}{\sqrt{2}}(d\theta_j - i d\theta_{j+1}) \\ d\eta_j^* &= \frac{1}{\sqrt{2}}(d\theta_j + i d\theta_{j+1}), \end{aligned}$$

such that

$$\int d\eta_j \eta_j = \int d\eta_j^* \eta_j^* = 1$$

is satisfied. Therefore

$$\int d\eta_j d\eta_j^* (-\eta_j \eta_j^*) = 1 \quad (57)$$

and more generally for the multivariate case [44, 45]

$$\int \prod_{j=1}^k d\eta_j d\eta_j^* \exp(-\eta_j A_{ji} \eta_i^*) = \det A. \quad (58)$$

But there is another point of view. We can consider  $\eta_j$  and  $\eta_j^*$  to be two independent generators of a complex Grassmann algebra. We again define

$$\int d\eta_j \eta_j = 1 \quad (59)$$

but this implies

$$\int d\eta_j^* \eta_j^* = -1 \quad (60)$$

using the star properties from Def. 4. It follows that

$$\int d\eta_j d\eta_j^* (\eta_j \eta_j^*) = 1 \quad (61)$$

contrary to Eq. (57).

What happens for the grade involution we are using in this paper? Our understanding of complex conjugated Grassmann variables is arguably more inclined to see  $\eta$  and  $\eta^\#$  as two independent variables. Even though we have no equivalent of Eqs. (59) and (60) (and we don't need it since we evaluate even Grassmanns only) we can define

$$\int d\eta_j d\eta_j^\# (\eta_j \eta_j^\#) \stackrel{\text{df}}{=} 1. \quad (62)$$

**Example.** Let's take the lowest dimensional case  $N = 2$  and calculate the modified Rogers norm of  $\tau = p_G(\varphi(q), \psi(p)) = 1 - (p - q)^2 \eta \eta^\#$ :

$$\begin{aligned} |\tau|_R &= \int d\eta d\eta^\# (1 + \eta \eta^\#) (1 - (p - q)^2 \eta \eta^\#) = \int d\eta d\eta^\# \eta \eta^\# - \int d\eta d\eta^\# \eta \eta^\# (p - q)^2 \\ &= 1 - (p - q)^2 = p(\varphi(q), \psi(p)). \end{aligned} \quad (63)$$

We got rid of Grassmann variables but another problem has appeared. The transition probability between two completely general pure supequbits reads

$$p(\varphi(q, \gamma, \delta), \psi(p, \alpha, \beta)) = (\alpha \bar{\gamma} + \beta \bar{\delta})(\bar{\alpha} \gamma + \bar{\beta} \delta)(1 - (p - q)^2). \quad (64)$$

The product  $(\alpha \bar{\gamma} + \beta \bar{\delta})(\bar{\alpha} \gamma + \bar{\beta} \delta)$  has its origin in the  $SU(2)$  subgroup of Eq. (30). The rest of Eq. (64) is the consequence of the other subgroup isomorphic to the group  $(\mathbb{R}, +)$  (the reals with addition) represented by the  $S$  matrix. We promised a solution after Lemma 15 to the apparent non-compact character of this group and in Eq. (64) we see why it is indeed a problem. There exists a choice of  $p$  and  $q$  such that the transition probability becomes negative. The probability function is meaningful only for  $0 \leq p(\varphi, \psi) \leq 1$  implying  $|p - q| \leq 1$ . This region is depicted on the left side of Fig. 1.

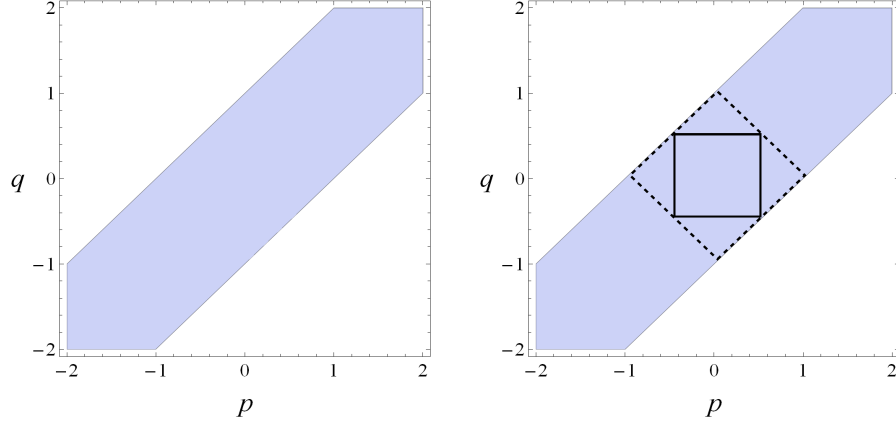


FIG. 1: The infinite blue stripe is the set where the transition probability between these two superqubits lies between zero and one. The two rectangles on the right indicate two subsets of  $\mathbb{R} \times \mathbb{R}$  as candidates of how to compactify the superqubit space. The dashed rectangle is the set  $s_1$  defined in Eq. (65). The inner rectangle is the subset  $s_2$  (Eq. (66)) motivated by Def. 25 dealing with the properties of transition probability functions.

Note that the probability of measurement of a superqubit  $\psi(p, \alpha, \beta)$  in the canonical basis Eq. (36) is reasonable for  $0 \leq |p| \leq 1$ . This motivates the following subset of allowed states

$$s_1 = \{p, q \in \mathbb{R}; |p - q| \leq 1 \cap |p + q| \leq 1\}. \quad (65)$$

The  $s_1$  is the rectangle demarcated by the dashed line on the right side of Fig. 1. This choice is not satisfactory though. If we set  $1/2 \leq |p| \leq 1$  for the measured state, then there exists a rotation of the canonical basis (in particular by  $S(2q\eta)$  with  $1/2 \leq |q| \leq 1$ ) such that the probability is negative again. In other words, for a given state it makes sense to talk about measurement in one basis but not in a rotated one. This is conceptually hard to accept and to avoid this problem we further restrict the set  $s_1$  to

$$s_2 = \{p, q \in \mathbb{R}; |p| \leq 1/2 \cap |q| \leq 1/2\}. \quad (66)$$

The set  $s_2$  is motivated by the following definition.

**Definition 25** (Physical states). *Let two superqubits  $\psi(p)$  and  $\varphi(q)$  satisfy  $0 \leq |p_G(\varphi(q), \psi(p))|_R \leq 1$ . The states are considered physical only for such  $p, q$  also satisfying*

$$0 \leq |p_G(\varphi(\pm q), \psi(\mp p))|_R \leq 1. \quad (67)$$

The definition ensures that for all  $p$  there will be  $q$  from the same interval such that the transition probability between the corresponding states lies between zero and one. This naturally introduces a Cartesian product  $P = P_\psi \times P_\varphi$  of two positivity domains  $P_\psi$  and  $P_\varphi$ . The positivity domain  $P_\psi \subset D_\psi$  where the set  $D_\psi$  is defined as

$$D_\psi = \{p \in \mathbb{R}; 0 \leq |p_G(|\psi(p)\rangle, |i\rangle)|_R \leq 1\}.$$

Def. 25 leads to

$$P_\psi = \{p \in \mathbb{R}; -1/2 \leq p \leq 1/2\}$$

and similarly for  $P_\varphi$ . This is consistent with the set  $s_2$  in Eq. (66) and the set is outlined by the solid rectangle in Fig. 1 (on the right).

We have cut a closed and bounded subset from  $\mathbb{R}$  where our super evolution is allowed to take place and this amounts to compactifying the original superqubit space – the sets  $P_\psi, P_\varphi$  are compact manifolds with boundary. Another virtue of Def. 25 is that for every  $S(2p\eta)$  there exists  $S^\dagger(2p\eta)$ . This is because  $S^\dagger(2p\eta) = S^{-1}(2p\eta) = S(-2p\eta)$  as we have noticed in Lemma 15. But not all group axioms are satisfied after we restricted the superqubit evolution to  $P_\psi$ . We know from Lemma 15 that  $S(2p_1\eta)S(2p_2\eta) = S(2(p_1 + p_2)\eta)$  but what if  $|p_1 + p_2| > 1/2$ ? The group law of addition is not defined beyond the domain  $P_\psi$ . Here we propose a solution based on the fact that  $(\mathbb{R}, +)$  is a universal cover of the compact group  $U(1)$ . The explicit onto map is the modulo  $2\pi$  function  $\text{mod } 2\pi : (\mathbb{R}, +) \mapsto U(1)$  that can be written as

$$p \text{ mod } 2\pi = p - 2\pi \left\lfloor \frac{p}{2\pi} \right\rfloor \quad (68)$$

valid for all  $p \in \mathbb{R}$ . If we make the following substitution

$$p \mapsto \frac{p}{2\pi} - \left\lfloor \frac{p}{2\pi} \right\rfloor - \frac{1}{2} \quad (69)$$

in Eq. (39) we obtain a superqubit with the right properties.

*Remark.* Perhaps there is a question why we bothered with Def. 25 if now we again compactified the whole  $\mathbb{R}$ . Def. 25 helped us to find where exactly we have to impose the periodic boundary conditions. If we imposed the periodic boundary conditions on the positivity interval leading to  $s_1$  we would encounter various inconsistencies [54].

*Remark.* The mapping Eq. (68) is a textbook example of a quotient space construction [48]. What makes it less trivial here is the presence of additional structures on the manifold we compactify.

One of the consequences of Def. 25 is that we cannot vary the parameter  $p$  such that the probability of measurement of the bullet state is one (note that before we bounded  $p$  the probability of measuring bullet had been one for  $p = \pm 1$ ). But this becomes more acceptable in the light of our earlier observation that the superqubit space is not a homogeneous space.

**Lemma 26.** *The superqubit compactification Eq. (68) is basis-independent.*

*Proof.* Up to now, we worked in a specific basis  $\{|0\rangle, |1\rangle, |\bullet\rangle\}$  but the compactification procedure should be independent on the basis. Let's see what happens if we transform a superqubit Eq. (39) into a rotated basis given by  $\{Z|0\rangle, Z|1\rangle, Z|\bullet\rangle\}$  where  $Z = U(\alpha, \beta)S(2p\eta)$  is an arbitrary  $UOSP(1|2)$  rotation. The group action followed by the change of Grassmann variables transforms  $|0\rangle$  to  $|\psi\rangle$  from Eq. (41). If compared to the  $S(2p\eta)$  subgroup acting on  $|0\rangle$  and followed by  $p \mapsto -p$  one gets almost an identical state

$$S(2p\eta)|0\rangle = \left(1 - \frac{p^2}{2}\eta\eta^\# \right) |0\rangle - p\eta |\bullet\rangle.$$

Only the action of  $SU(2)$  is left out but that is confined to the even subspace and therefore is not relevant for the proof. So we can just study the effect of the rotated canonical basis  $\{|0'\rangle = S(2x\eta)|0\rangle, |1'\rangle = S(2x\eta)|1\rangle, |\bullet'\rangle = S(2x\eta)|\bullet\rangle\}$  where  $x \in \mathbb{R}$ . We rewrite the transformed superqubit as

$$|\psi'\rangle = \left(1 - \frac{(p-x)^2}{2}\eta\eta^\# \right) |0'\rangle - (p-x)\eta |\bullet'\rangle.$$

We want this state to be a physical state according to Def. 25 and so we impose  $|p'| \leq 1/2$  where  $p - x = p'$ . But this is not enough and the argument now goes exactly as in the paragraph leading to Eq. (68) – the compactification in the new basis is achieved by the same prescription as Eq. (69)  $p' \mapsto \frac{p'}{2\pi} - \left\lfloor \frac{p'}{2\pi} \right\rfloor - \frac{1}{2}$ . ■

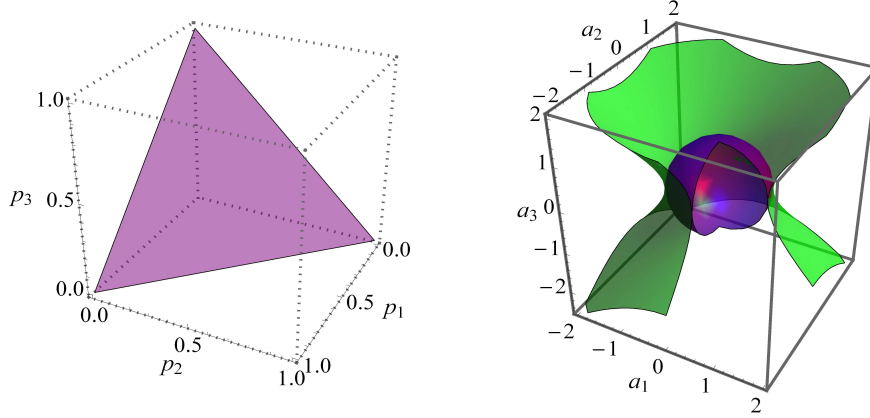


FIG. 2: The probability simplex  $\sum_{i=0}^2 p_i = 1$  (on the left) can be viewed as a space of probability distribution  $\sum_{i=0}^2 a_i^2 = 1$  for a real qutrit (the purple sphere on the right). In the superqubit case,  $a_2$  becomes  $a_\bullet$  and we effectively have two surfaces  $a_0^2 + a_1^2 \pm a_\bullet^2 = 1$ . So before the compactification procedure is performed the superqubit space of probability distribution consists of the sphere and a hyperboloid (the green surface on the right).

### Summary of Section II

Let's conclude by summarizing of what we have done in this section and why. Unlike ordinary quantum mechanics, it is not sufficient to define a Hilbert space for transition probabilities to lie between zero and one. The definition of a Hilbert space only guarantees the existence of states whose norm is one. But when transition probabilities are calculated they are, in general, even Grassmann supernumbers. To extract real numbers we constructed in [15] the modified Rogers norm. But the norm transforms even Grassmann-valued probabilities into numbers that may lie outside the interval  $[0, 1]$ .

Let's compare a superqubit with a real three-level quantum system  $|v\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle$  (a real qutrit where  $a_i \in \mathbb{R}$ ). The probability of measurement  $|i\rangle$  is  $p_i = a_i^2$  and of course  $0 \leq p_i \leq 1$  and  $\sum_i p_i = 1$  since the qutrit is normalized. A simple observation  $\sum_i p_i = \sum_i a_i^2$  is a starting point of statistical geometry [49] – a remarkable insight of differential geometry into probability theory and information theory. One the left side of the equation we have a probability simplex but on the right side we recognize the  $S^2$  sphere.

*Remark (important).* Note that we have to distinguish between the superqubit manifold (that is, where  $UOSP(1|2)$  acts) brought in Def. 13 and the space of superqubit probability distribution informally introduced for the purpose of this summary.

In the superqubit case the situation is different. Superqubits are also bound to the hyperplane  $\sum_i p_i = 1$  but before we compactify them the constraints on the probabilities are weaker:  $p_0 \leq 1, p_1 \leq 1$  and  $p_2 \equiv p_\bullet \in \mathbb{R}$  (cf. Eq. (41)). From the geometrical point of view we are able to jump from the sphere  $a_0^2 + a_1^2 + a_\bullet^2 = 1$  to the hyperboloid  $a_0^2 + a_1^2 - a_\bullet^2 = 1$  by tuning the parameter  $p$  in Eq. (41). We illustrate the situation in Fig. 2. On the left, there is a probability simplex. On the right, the purple sphere is the space of the qutrit probability distribution together with the green hyperboloid the superqubit is extended to. After the proposed compactification of the superqubit manifold, the superqubit probability space will become only the sphere. Note that the compactification procedure can be achieved ‘manually’ – just by requiring  $p_0, p_1, p_\bullet \in [0, 1]$ .

In [15] we argued why superqubits are unlike  $SU(2, 1)$  ‘qutrits’. In Fig. 2 we can visualize the difference. The probability space of the  $SU(2, 1)$  qutrits is only the hyperboloid  $a_0^2 + a_1^2 - a_2^2 = 1$ . The sphere can be obtained by the following compactification:  $a_2 \mapsto ia_2$ .

### III. BIPARTITE SUPERQUBIT STATES, THE CHSH GAME AND TSIRELSON’S BOUND

The most interesting results of quantum information theory are when bi- and multipartite states are used as resources in computational and communication protocols. Quantum correlations are the distinctive aspect of quantum physics and one of the consequences is that using multipartite entangled quantum states one can perform significantly better compared to classical physics. Here we want to argue that multipartite entangled quantum states based on superqubits are even better resources than ordinary quantum states. But we face an obstacle. It is not immediately obvious what is the Lie superalgebra one should study. Moreover, the representation theory of higher-dimensional Lie superalgebras is not straightforward [16, 18]. We will follow a different path here. Using our definition of a Hilbert space (Def. 17) we conjecture the existence of certain states (three examples in particular) that we have good reasons to think that they are members of the carrier space of the Grassmann-valued group we would have obtained by studying higher orthosymplectic Lie superalgebras. The first example is a ‘sure thing’. Whatever algebra there exists for two superqubits it must admit a tensor product of two superqubits. Let’s utilize the transformed superqubits from Eq. (41):

$$\begin{aligned} |\psi\rangle_A |\psi\rangle_B = & \left(1 - \frac{p_A^2}{2} \eta_A \eta_A^\# \right) \left(1 - \frac{p_B^2}{2} \eta_B \eta_B^\# \right) (\alpha |0\rangle + \beta |1\rangle) (\gamma |0\rangle + \delta |1\rangle) \\ & + p_B \eta_B \left(1 - \frac{p_A^2}{2} \eta_A \eta_A^\# \right) (\alpha |0\bullet\rangle + \beta |1\bullet\rangle) + p_A \eta_A \left(1 - \frac{p_B^2}{2} \eta_B \eta_B^\# \right) (\gamma |\bullet 0\rangle + \delta |\bullet 1\rangle) \\ & + p_A p_B \eta_A \eta_B |\bullet\bullet\rangle, \end{aligned} \quad (70)$$

where  $p_A, p_B \in \mathbb{R}$ ,  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$  and  $\eta_A, \eta_A^\#, \eta_B, \eta_B^\# \in \mathcal{C}\Lambda_4$ .

As expected, the state  $|\bullet\bullet\rangle$  does not appear accompanied by ordinary numbers as a consequence of Lemma 14. Hence, we propose a general form (not necessarily the most general form) of a pure two-superqubit state to be

$$\begin{aligned} \Psi_{AB} = & \left(1 + \frac{X}{2} + \frac{3}{8} X^2 \right) (a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle) \\ & + p_B \eta_B \left(1 - \frac{p_A^2}{2} \eta_A \eta_A^\# \right) (\alpha |0\bullet\rangle + \beta |1\bullet\rangle) + p_A \eta_A \left(1 - \frac{p_B^2}{2} \eta_B \eta_B^\# \right) (\gamma |\bullet 0\rangle + \delta |\bullet 1\rangle) \\ & + p_A p_B \eta_A \eta_B |\bullet\bullet\rangle, \end{aligned} \quad (71)$$

where  $p_A, p_B \in \mathbb{R}$ ,  $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{C}$  such that  $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$ ,  $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$  and

$$X = -p_A^2 \eta_A \eta_A^\# - p_B^2 \eta_B \eta_B^\# - p_A^2 p_B^2 \eta_A \eta_A^\# \eta_B \eta_B^\#.$$

We rewrite

$$1 + \frac{X}{2} + \frac{3}{8} X^2 = \left(1 - \frac{p_A^2}{2} \eta_A \eta_A^\# \right) \left(1 - \frac{p_B^2}{2} \eta_B \eta_B^\# \right).$$

The state  $\Psi_{AB}$  contains an arbitrary two-qubit state and so  $su(4)$  would be a subalgebra of the studied structure. The matrix of the invariant sesquilinear form would be  $G \otimes G$  with  $G$  from



Eq. (46) according to which it is normalized to one. Let's set  $a = d = 1/\sqrt{2}$  and  $\beta = \delta = 1$  and we obtain the state we are going to experiment with:

$$\begin{aligned} \Upsilon_{AB}(p_A, p_B) = & \left(1 - \frac{p_A^2}{2} \eta_A \eta_A^\# \right) \left(1 - \frac{p_B^2}{2} \eta_B \eta_B^\# \right) \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\ & + p_B \eta_B \left(1 - \frac{p_A^2}{2} \eta_A \eta_A^\# \right) |1\bullet\rangle + p_A \eta_A \left(1 - \frac{p_B^2}{2} \eta_B \eta_B^\# \right) |\bullet 1\rangle + p_A p_B \eta_A \eta_B |\bullet\bullet\rangle. \end{aligned} \quad (72)$$

We can claim that  $\Upsilon_{AB}$  is at least as nonlocal as a maximally entangled (Bell) state. If we prepare any setup where a maximally entangled state is used in quantum information theory, utilize  $\Upsilon_{AB}$  instead and ignore the bullet components ( $p_A = p_B = 0$ ) we will be able to perform as efficiently as with the Bell state itself. The question is now: Is  $\Upsilon_{AB}$  able to perform better considering the super degrees of freedom? The best way to check is to reproduce the experiment that is a hallmark of nonlocality – the coincidence measurement resulting in Bell's inequalities [1]. There exists a sharp reformulation of Bell inequalities known as the CHSH game [3] interpreting the measurement from the computer science point of view. Let us recapitulate the CHSH game. It is a so-called nonlocal game [5] with three players: a referee who competes with two cooperating players Alice and Bob. The referee chooses two bits  $i \in \{0, 1\}$  and  $j \in \{0, 1\}$  with probability  $1/4$  and sends  $i$  to Alice and  $j$  to Bob such they are not aware of one another's bit value. Alice and Bob each return a bit of communication (denoted  $a$  and  $b$ , respectively) back to the referee. The condition for Alice and Bob to win the game is when the equation  $ij = a \oplus b$  is satisfied for each round.

Alice and Bob cannot communicate during the game but they can establish their strategy beforehand. They also share a resource – a physical system obeying the known laws of physics. The agreed strategy can be looked upon as a type of classical resource (classical correlations). In that case, the optimal strategy leads to the maximal probability of winning

$$p_{win}^{class} = \frac{3}{4}.$$

If they share quantum correlations the chances of winning are higher. Namely, a shared maximally entangled state  $\Psi_{AB} = 1/\sqrt{2}(|00\rangle + |11\rangle)$  accompanied by an agreed measurement strategy leads to

$$p_{win}^{quant} = \cos^2 \frac{\pi}{8} \simeq 0.8535.$$

As a matter of fact, this is the maximal value that can be reached for the CHSH game using quantum-mechanical resources. It is known as Tsirelson's bound [2]. To achieve the bound they choose one of the following orthogonal measurement bases  $\{|0\rangle_{iA}, |1\rangle_{iA}\}$  and  $\{|0\rangle_{jB}, |1\rangle_{jB}\}$  rotated according to the value they receive from the referee  $\{i, j\} \rightarrow \{\alpha_i, \beta_i, \gamma_j, \delta_j\}$  where

$$\begin{aligned} |0\rangle_{iA} &= \alpha_i |0\rangle_A + \beta_i |1\rangle_A \\ |0\rangle_{jB} &= \gamma_j |0\rangle_B + \delta_j |1\rangle_B \end{aligned}$$

and similarly for  $|1\rangle_{iA(jB)}$ . The amplitudes achieving Tsirelson's bound read

$$\begin{aligned} (ij = 00) &\rightarrow \{\alpha_0 = 1, \beta_0 = 0, \gamma_0 = \cos \frac{\pi}{8}, \delta_0 = \sin \frac{\pi}{8}\} \\ (ij = 01) &\rightarrow \{\alpha_0 = 1, \beta_0 = 0, \gamma_1 = \cos \frac{\pi}{8}, \delta_1 = -\sin \frac{\pi}{8}\} \\ (ij = 10) &\rightarrow \{\alpha_1 = \frac{1}{\sqrt{2}}, \beta_1 = \frac{1}{\sqrt{2}}, \gamma_0 = \cos \frac{\pi}{8}, \delta_0 = \sin \frac{\pi}{8}\} \\ (ij = 11) &\rightarrow \{\alpha_1 = \frac{1}{\sqrt{2}}, \beta_1 = \frac{1}{\sqrt{2}}, \gamma_1 = \cos \frac{\pi}{8}, \delta_1 = -\sin \frac{\pi}{8}\}. \end{aligned}$$

Up until now there has been no candidate among physical theories that could provide resources more nonlocal than a maximally entangled state. The only possibility is a nonlocal box (also called PR box) [4] as a mathematical construct designed to reach the maximal winning probability  $p_{win}^{PR} = 1$ . A nonlocal box is a hypothetical resource shared by Alice and Bob whose inputs are  $i$  and  $j$  and its highly nonlocal inner workings produce the values  $a$  and  $b$  such that Alice and Bob always win.

If we want to test how well  $\Upsilon_{AB}$  performs we have to adjust the rules of the CHSH game but at the same time we have to play exactly the same game as we play with a Bell state. A superqubit is formally a three-level system and so we merge the subspace spanned by  $|1\rangle$  and  $|\bullet\rangle$ . We set the rules such that Alice (Bob) announces the result  $a = 1$  ( $b = 1$ ) if the result of the measurement lies in this subspace and  $a = 0$  ( $b = 0$ ) if it was projected into  $|0\rangle$ . We define

$$Z_{iA} \otimes Z_{jB} = S(2r_i \eta_A) U(\alpha_i, \beta_i) \otimes S(2s_j \eta_B) U(\gamma_j, \delta_j), \quad (73)$$

where  $\eta_A, \eta_A^\#$ ,  $\eta_B$  and  $\eta_B^\#$  are generators of the Grassmann algebra  $\mathcal{C}\Lambda_4$  and  $r_i, s_j \in \mathbb{R}$  is chosen according to the bits  $i$  and  $j$  received from the referee. The local superunitary transformation is a general rotation Eq. (30).

The measurement will be performed on a shared bipartite entangled superqubit state  $\Upsilon_{AB}$  rotated according to Eq. (73)

$$\Upsilon_{iA,jB} = (Z_{iA} \otimes Z_{jB}) \Upsilon_{AB}. \quad (74)$$

Therefore the winning Grassmann-valued probability reads

$$p_{Gwin}(\Upsilon_{AB}) = \frac{1}{4} \sum_{ij \in \{00,01,10\}} \left( p_{G00}^{(ij)} + p_{G11}^{(ij)} + p_{G1\bullet}^{(ij)} + p_{G\bullet 1}^{(ij)} + p_{G\bullet\bullet}^{(ij)} \right) + p_{G01}^{(11)} + p_{G10}^{(11)} + p_{G0\bullet}^{(11)} + p_{G\bullet 0}^{(11)}, \quad (75)$$

where

$$p_{Gmn}^{(ij)} = \langle m_A n_B | \Upsilon_{iA,jB} \rangle \langle \langle m_A n_B | \Upsilon_{iA,jB} \rangle \rangle^\# \quad (76)$$

is the Grassmann-valued probability function introduced in Def. 20. The letters  $m$  and  $n$  label the orthogonal basis states  $\{|m\rangle, |n\rangle\} = \{|\bullet\rangle, |0\rangle, |1\rangle\}$ .

**Theorem 27.** *The state  $\Upsilon_{AB}$  from Eq. (72) used as a resource in the CHSH game with the restrictions on physical state (Def. 25) crosses Tsirelson's bound reaching  $p_{win}^{sqbit} \simeq 0.8647$ .*

*Remark.* Note the difference between  $\Upsilon_{AB}$  and  $\Gamma_{AB}$  studied in [15].

*Proof.* We define

$$p_{win} = \max_{\substack{p_A, p_B, r_i, s_j \\ \alpha_i, \beta_i, \gamma_j, \delta_j}} p_{win}(\Upsilon_{AB}) \quad (77a)$$

$$\text{s.t. } |r_i| \leq 1/2, |s_j| \leq 1/2, \quad (77b)$$

$$|p_A| \leq 1/2, |p_B| \leq 1/2, \quad (77c)$$

$$0 \leq p_{mn}^{(ij)} \leq 1, \quad \forall i, j, m, n, \quad (77d)$$

where  $p_{win}(\Upsilon_{AB})$  is Eq. (75) after the modified Rogers norm from Def. 24 has been used. The constraint in Eq. (77b) is Def. 25 applied on a tensor product of two superqubits  $\psi(r_i)$  and  $\varphi(s_j)$  (cf. Eq. (55)). The constraint in Eq. (77c) follows from Def. 25 applied on  $\Upsilon_{AB}$ . It is surprisingly equivalent to the previous constraint since the transition probability factorizes

$$p_G(\Upsilon_{AB}(p_A, p_B), \Upsilon_{AB}(q_A, q_B)) = \left(1 - (p_A - q_A)^2 \eta_A \eta_A^\#\right) \left(1 - (p_B - q_B)^2 \eta_B \eta_B^\#\right) \\ \xrightarrow{\text{Def. 24}} (1 - (p_A - q_A)^2) (1 - (p_B - q_B)^2).$$

The third line is a constraint that expresses our ignorance about how to get rid of negative probabilities for the measurement of  $\Upsilon_{AB}$  in an arbitrary, locally superrotated, basis. The simple procedure from Def. 25 followed by the compactification must be generalized. The reason is that there is no factorization happening for

$$\langle m_A n_B | (Z_{iA} \otimes Z_{jB}) | \Upsilon_{AB}(p_A, p_B) \rangle$$

for an arbitrary rotation  $Z_{iA} \otimes Z_{jB}$ . These are the expressions forming the transition probability  $p_{Gmn}^{(ij)}$  of a general projective measurement Eq. (76) used for the calculation of the winning probability. So there does not seem to exist a sole condition on the  $p_A, p_B$  parameters to get positive probabilities – they are intertwined with the parameters  $\alpha_i, \beta_i, \gamma_j$  and  $\delta_j$  coming from the  $SU(2)_A \otimes SU(2)_B$  subgroup.

Hence, we have no equivalent of Lemma 26 for single superqubits and Eqs. (77b) and (77c) are not sufficient to guarantee the positivity of the transition probabilities. It must be enforced ‘manually’ as in Eq. (77d). We illustrated this procedure on the single superqubit space of probability distribution in the summary of Sec. II (cf. Fig. 2). This step is crude but if a consistent compactification is in principle possible even for two superqubits (that is an open question), it will lead to the same result – a two-superqubit Hilbert space devoid of any pathological effects such as negative transition probabilities. However, the two-superqubit manifold will likely be a non-trivial surface whose compactification might not be straightforward.

Note that we require all thirty six transition probabilities to lie between zero and one since the losing probabilities can be in principle measured if Alice and Bob, for some reason, decide to do so.

The overall expression for  $p_{win}$  is complicated and its form is not really informative. The optimization has to be done numerically [50] and gives us  $p_{win}^{sqbit} \simeq 0.8647$  with the following winning parameters:

$$\begin{aligned} p_A &\simeq 1/2, \quad p_B \simeq 0 \\ r_0 &\simeq -0.3445, \quad s_0 \simeq 0, \quad r_1 \simeq 0.3449, \quad s_1 \simeq 0 \\ \alpha_0 &\simeq 0.2066, \quad \alpha_1 \simeq -0.2090, \quad \beta_0 \simeq 0, \quad \beta_1 \simeq -2.3572. \end{aligned}$$

The optimization procedure leads to a non-convex program and so  $p_{win}^{sqbit} \simeq 0.8647$  is not necessarily a global maximum. ■

#### IV. CONCLUSIONS

In this work we studied superqubits – an extension of quantum mechanics based on supersymmetry [14]. The motivation was to properly define the mathematical structures used in [14, 15] and offer a way of getting rid of negative probabilities encountered in [15]. This has been achieved by the proposed compactification procedure resolving the problem for single superqubits. We also argued here that the presented theory does not modify quantum mechanics but rather extends it into a specific supersymmetric domain. If we ‘switched off’ the super degrees of freedom we would get quantum mechanics back unscathed.

We started by recalling some basic facts from the theory of Lie superalgebras and superlinear algebra. Following the introduced formalism we studied the algebraic properties of superqubits in detail. In the last section we ventured into the territory of multi-superqubit states and constructed bipartite superentangled states. One such state (a different one from the state used in [15]) was used as a nonlocal resource in a three-party game known as the CHSH game. The game is a perspicuous reformulation of the CHSH inequalities from the quantum communication complexity theory point of view. The best performance quantum mechanics is capable of is when

a maximally entangled state is used as a shared nonlocal resource in the game between Alice and Bob. The maximum winning probability is then  $p_{win}^{quant} = \cos^2 \pi/8 \simeq 0.8536$  which in terms of an expected value of an operator corresponds to so-called Tsirelson's bound [2]. It has been known, however, that quantum mechanics is not as nonlocal as it could have been. There exists a gap beyond Tsirelson's bound filled with hypothetical no-signalling theories (that is, theories not permitting superluminal communication) but more nonlocal than quantum mechanics. In [15] we reported crossing Tsirelson's bound using a concrete physical model based on superqubits. Here, due to the introduced compactification procedure, we further limited the parameter space of superqubits while still being able to cross the bound. The maximal winning probability we found is lower compared to [15]:  $p_{win}^{sqbit} \simeq 0.8647$ .

This study leaves several questions unanswered. First of all, how else are superqubits different from quantum mechanics? Or, even more generally, does this theory fit into the framework of general probabilistic theories studied recently by a number of authors [51–53]? It might be of interest to see if all desirable axioms are satisfied and, if not, what the consequences would be. After all, the version of supersymmetric quantum mechanics we set out to explore possibly extends quantum mechanics even without crossing Tsirelson's bound. Even if Tsirelson's bound was not beaten we would still be left with states that are unlike ordinary quantum-mechanical states. This brings us to another question. How can we get rid of negative probabilities for bipartite, and possibly multipartite states? Negative probabilities are never used to calculate anything but the theory is still incomplete since they can be reached by the group action followed by the modified Rogers norm. We believe that the compactification procedure introduced here can be generalized for multisuperqubit states. The answer how to achieve this goal certainly lies on the way to the proper definition of a Grassmann-valued group governing the evolution of multipartite superqubits. That is a research project on its own that we avoided and instead used a dirty way to get around the problem in Section III by using the insight from the theory of superqubits obtained in the first two sections. Finally, in the previous work [15] we defined the modified Rogers norm as a way how to extract real numbers from even Grassmann number. This is by no means a unique procedure. It might be interesting to propose and study alternative prescriptions.

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